

From Bi-immunity to Absolute Undecidability

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Abstract

An infinite binary sequence A is absolutely undecidable if it is impossible to compute A on a set of positions of positive upper density. Absolute undecidability is a weakening of bi-immunity. Downey, Jockusch and Schupp [2] asked whether, unlike the case for bi-immunity, there is an absolutely undecidable set in every non-zero Turing degree. We provide a positive answer to this question by applying techniques from coding theory. We show how to use Walsh-Hadamard codes to build a truth-table functional which maps any sequence A to a sequence B , such that given any restriction of B to a set of positive upper density, one can recover A . This implies that if A is non-computable, then B is absolutely undecidable. Using a forcing construction, we show that this result cannot be strengthened in any significant fashion.

1 Introduction

Let $A \in 2^\omega$ be a non-computable infinite binary sequence. How can we gauge the extent to which A is close to being computable? Many basic concepts in computability theory can be seen as providing a partial answer to this question. For example, if A is low, Δ_2^0 , or of minimal degree, then A could be seen as being close to computable.

In this paper, we will consider a different perspective. Given A and a partial computable function φ we will say A *extends* φ if for all $x \in \text{dom}(\varphi)$, $A(x) = \varphi(x)$ (where $A(n)$ is the value of the n -th bit of A). We will measure how close A is to being computable by considering the size of the domain of any partial computable function that A extends (the larger the domain, the closer A is to being computable).

A set A is bi-immune if neither A nor its complement contain an infinite c.e. (or, equivalently, computable) subset. It is not difficult to see that A is bi-immune if and only if A does

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not extend a partial computable function with infinite domain. Hence, from this paper’s perspective, bi-immune sets are as far from computable as possible. Once bi-immune sets are shown to exist, the property of being bi-immune is clearly not invariant under Turing equivalence. However, it does make sense to ask whether every non-zero Turing degree contains a bi-immune set. The answer is no.

Theorem 1 (Jockusch [4]). *There exists a non-zero Turing degree such that no element of this degree is bi-immune.*

To determine whether a sequence A is bi-immune, we asked whether A extends a partial computable function with infinite domain. Essentially, bi-immunity regards the domain of a partial computable function as large if it is infinite. We can weaken the notion of bi-immunity by strengthening our concept of largeness.

Definition 2. Let $D \subseteq \mathbb{N}$. The *upper density* of D is the quantity

$$\rho(D) := \limsup_{n \rightarrow \infty} \frac{|D \cap \{0, \dots, n-1\}|}{n}.$$

When the upper density of D is zero, we simply say that D has *density 0*.¹

Definition 3 (Myasnikov-Rybalov [6]). We say that $A \in 2^\omega$ is *absolutely undecidable* if there is no partial computable function $\varphi : \mathbb{N} \rightarrow \{0, 1\}$ whose domain has positive upper density and such that $\varphi(n) = A(n)$ for all n in the domain of φ .

The notion of absolute undecidability comes from the study of generic computability [2, 3, 5]. A sequence $A \in 2^\omega$ is generically computable if it can be “computed modulo a set of density 0”, i.e., if there is a partial computable function φ such that $\varphi(n) = A(n)$ for all n in the domain of φ , and $\rho(\mathbb{N} \setminus \text{dom}(\varphi)) = 0$. Absolute undecidability is the antithesis of generic computability; a set is absolutely undecidable if it cannot be computed at all, modulo a set of positions of density 0.

In Downey, Jockusch and Schupp [2], the following question is asked.

Question 4. *Does every non-zero Turing degree contain a set which is absolutely undecidable?*

In other words, does Theorem 1 fail if we weaken bi-immunity to absolute undecidability? We give a positive answer to this question in Theorem 5. In fact every non-zero *truth-table* degree contains a sequence which is absolutely undecidable.

Theorems 1 and 5 show that an interesting dichotomy exists between bi-immunity and absolute undecidability. In Section 4, of this paper we investigate what happens between these two notions.

To express this idea precisely, we need to introduce some definitions. For a function $h : \mathbb{N} \rightarrow \mathbb{R}^{>0}$, denote by $o(h)$ the set of functions $f : \mathbb{N} \rightarrow \mathbb{R}$ such that $\lim_{n \rightarrow \infty} f(n)/h(n) = 0$.

¹Density, defined as above but using \lim instead of $\lim \sup$, is often denoted by ρ . As density will not play a significant role in this paper we will simplify notation by using ρ to denote upper density instead of the more common $\bar{\rho}$.

In particular, write $o(n)$ for $o(g)$ when g is the function $n \mapsto n$. Given $D \subseteq \mathbb{N}$, define $\kappa_D: n \mapsto |D \cap \{0, \dots, n-1\}|$ and $\rho_D: n \mapsto \kappa_D(n)/n$. Theorem 5 is a uniform version of the following statement.

Every non-zero truth-table degree contains an element X such that if φ is a partial computable function and X extends φ then $\kappa_{\text{dom}(\varphi)} \in o(n)$.

To determine what happens between bi-immunity and absolute undecidability, we can ask whether this statement can be strengthened by replacing $o(n)$ with a smaller set of functions. In particular, is this statement still true if we replace $o(n)$ with $o(h)$ for some computable function $h \in o(n)$ e.g. $n/\log \log n$?

We will show that the statement cannot be strengthened in this manner. If h is a computable function and $h(n) \in o(n)$, then there is a non-zero Turing degree \mathbf{a} such that for all sets $X \leq_T \mathbf{a}$, X extends a partial computable function g with $\kappa_{\text{dom}(g)} \notin o(h)$. In fact we will construct a single Turing degree that works for all such h . This is the content of Theorem 11. Hence the threshold where the behavior changes between bi-immunity and absolute undecidability occurs right at the absolute undecidability end.

Theorem 11 is proved using forcing with computable perfect trees. The structure of this proof is similar to that of Jockusch's proof of Theorem 1 [4], but requires some extra machinery to deal with density.

2 Coding Theory and Absolute Undecidability

Our objective in this section and the following section, is to answer Question 4 by proving the following theorem.

Theorem 5. *There exists a tt-functional Φ such that for every non-computable sequence $A \in 2^\omega$, Φ^A is absolutely undecidable and $\Phi^A \equiv_{tt} A$.*

Let us present the structure of our proof. We will show how to encode any sequence A via a tt-reduction Φ into a sequence B in such a way that A can be fully recovered (by another tt-reduction) given B on a set of positive upper density. If A is non-computable, then $B = \Phi^A$ is necessarily absolutely undecidable. Otherwise, B would extend a partial computable function φ with domain a set of positive upper density and thus A could be computed using φ .

Our proof will have an element of non-uniformity. Instead of reproducing the set A exactly (given B on a set of positive upper density) we will produce a computable tree of bounded width having A as a path. This is sufficient because any path through a computable tree of bounded width is itself computable. Theorem 10 shows that this non-uniformity cannot be avoided, and, in fact, that even any finite number of decoding procedures do not suffice to decode an arbitrary set A . This is shown to hold for an arbitrary coding scheme, not just for the specific scheme Φ used in the proof of Theorem 5.

The functional Φ is built upon an error correcting code known as a Walsh-Hadamard code (see, e.g. Arora and Barak [1, Section 19.2.2]). An error correcting code maps finite

strings to finite strings, thus for our purposes we will have to use the Walsh-Hadamard code iteratively. On input A , for all n , the initial segment (or prefix) of A of length n , which we denote by $A \upharpoonright n$, is encoded into a string σ_n . The image of A under Φ will be the sequence $B = \sigma_1 \sigma_2 \sigma_3 \dots$.

Ideally we would like a computable sequence of functions $\{f_n\}_{n \in \mathbb{N}}$, where f_n is a function from binary strings of length n to binary strings of some length k_n with the following property. For any $\delta > 0$, for all sufficiently large n , for all strings σ of length n , any $\lfloor \delta \cdot k_n \rfloor$ bits of $f(\sigma)$ uniquely determines σ . In coding theory, such an coding is called an *erasure code*: one encodes a string σ into $\tau = f(\sigma)$ in such a way that σ can be recovered from any subset τ' of the bits of τ of sufficiently large size (think of τ' as a version of τ where some of the bits are missing, say replaced with '?', but those that are given are correct).

If there were such a coding, we could attempt to prove Theorem 5 by defining B to be equal to

$$f_1(A \upharpoonright 1) f_2(A \upharpoonright 2) f_4(A \upharpoonright 4) f_8(A \upharpoonright 8) \dots$$

Now if the sequence k_n has the property that given B on a set of positive upper density δ , then for some constant $c > 0$, for infinitely many n , one must have $\lfloor \delta/c \cdot k_n \rfloor$ bits of $f_n(A \upharpoonright n)$, then we could reconstruct $A \upharpoonright n$ for these n , and hence recover A itself. While not all sequences k_n have this property, the sequence defined by letting $k_n = 2^n$ does (as we will prove in Lemma 8). This is the sequence we will make use of in our proofs.

Unfortunately, requiring that any $\lfloor \delta \cdot k_n \rfloor$ bits of $f(\sigma)$ uniquely determine σ as described above, is too much to ask for. There is a theoretical limit to recovering an encoded string unambiguously from a small fraction of the bits of its encoding.² For simplicity let us say that we only want to encode three different strings $\sigma_1, \sigma_2, \sigma_3$ into codewords τ_1, τ_2, τ_3 . Without loss of generality we can assume that the codewords all have the same length m . For each position $i \leq m$, at least two codewords agree on the bit number i . Thus, by the pigeonhole principle, there exist two strings among τ_1, τ_2, τ_3 that agree on at least one third of their bits.

This seems to defeat the plan of retrieving the original string from a small fraction of the bits of its encoding, as the simple example above shows that any fraction smaller than $1/3$ is insufficient in general. The solution to this problem is to use another concept from coding theory: *list decoding*. In the remainder of this section we will review these concepts and explain how they can be used to prove Theorem 5. However, we appreciate that some readers will not have a background in coding theory, so in the following section we will give a proof of Theorem 5 using basic facts about vector spaces, that assumes no prior knowledge of coding theory.

A *list decodable code* is an error correcting coding scheme such that, if the codeword z for a string x is known without too many errors, one can produce a small list of strings to which x belongs. Define the (normalized) Hamming distance between two words of equal length n as $d_H(x, y) := |\{i \mid x(i) \neq y(i)\}|/n$. The *minimum distance* of a coding scheme is the minimum normalized Hamming distance between two distinct codewords. List decodable codes ensure that a ball (in Hamming distance) around a potential codeword contains only a small number of actual codewords. This means that a potential codeword can be decoded

²The same problem occurs for any finite alphabet size.

into a small number of possible original strings. In our case this number will be constant, which will allow us to build a computable tree of bounded width.

The Walsh-Hadamard code, which we define in the next section, maps words of length n to codewords of length 2^n . In [1, Section 19.2.2] it is shown that this is a code with minimum distance $1/2$, i.e., any two codewords x, y disagree on at least half of their bits. Assume we have a “corrupted” version of a codeword that could either be x or y , where corrupted means that some bits b have been replaced by $1 - b$. Assume that we are guaranteed that the corruption has only happened on at most a quarter of all bits. Then we can be sure that fewer than half of the bits where x and y disagree have been corrupted, so we can take a majority vote on those bits to decide whether the given corrupted codeword came from x or y . This is completely insufficient for our purposes, of course, since in our construction we will in general only be guaranteed a much smaller fraction $\varepsilon \ll 1/4$ of bits of the codeword. But we can apply the following theorem:

Theorem 6 (Johnson bound (see, e.g., [1])). *If $E : \{0, 1\}^n \rightarrow \{0, 1\}^m$ is an error correcting code of minimum distance at least $1/2$, then for every $x \in \{0, 1\}^m$ and $\delta > 0$, there exist at most $l = 1/(2\delta^2)$ elements y of $\{0, 1\}^n$ such that $d_H(x, E(y)) \leq 1/2 - \delta$.*

In other words, if we have a corrupt version z' of a codeword $z = E(y)$, which is good enough in the sense that a fraction $1/2 + \delta$ of the bits of z' coincide with those of z , we can generate from z' a list of size $1/(2\delta^2)$ of possible candidates for y which will indeed contain y .

In our case, we do not have such a z' , but we have instead a known fraction of bits of z which we are sure are correct and we know their position in z . As we explained above, what we need for this situation is an erasure code.

The following argument shows that we can still use the Walsh-Hadamard code in this context. Imagine we know a fraction 2δ of the bits of a codeword z for the original word y , together with their respective positions in z . Now fill all the remaining positions with 0's to get a potential codeword z_0 and with 1's to get a potential codeword z_1 . Perform a list-decoding on both z_0 and z_1 , thus getting two lists of possible candidates of size at most $1/(2\delta^2)$. Merge the two lists; the resulting list has size at most $2 \cdot 1/(2\delta^2) = 1/\delta^2$. Note that the correct entry is on the list since either z_0 or z_1 must have a fraction at least $1/2 + \delta$ of its bits in common with z , so the list contains the original word y .

3 Every truth table degree contains an absolutely undecidable set

We hope that the reader who is familiar with coding theory can find in the previous section sufficient information to prove Theorem 5. In this section, we present our argument in a self-contained way that assumes no prior knowledge of coding theory. Besides, the direct analysis of the Walsh-Hadamard code for our purposes will give a better bound on the list of candidates than the one we gave in the previous section ($1/\delta$ instead of $1/\delta^2$).

The core of the proof resides in a combinatorial argument about vector spaces. Before giving the argument let us illustrate it with an example. Define $f_3 : \{0, 1\}^3 \rightarrow \{0, 1\}^7$ as

follows. If $a, b, c \in \{0, 1\}$ then

$$f_3(abc) = abc(a+b)(b+c)(a+c)(a+b+c),$$

where addition is performed modulo 2 (caveat: the right-hand side is a concatenation of bits, not a product modulo 2). In other words, $f_3(abc)$ is obtained by concatenating together the output from all linear functionals from $\{0, 1\}^3$ to $\{0, 1\}$ with argument abc . (Note that we do not include the zero functional in this example.) Given any $a, b, c \in \{0, 1\}$ assume we are given 3 bits of $f_3(abc)$, e.g. the zeroth bit is 1, the fourth bit is 0 and sixth bit is 1. The zeroth bit tells us that $a = 1$, the fourth bit tells us that $b + c = 0$, the sixth bit is redundant information because we already know that $a + b + c = 1$ from the zeroth and fourth bit. Nevertheless, we can determine that $abc \in \{100, 111\}$ i.e. we know that abc is one of two possibilities. Further, no matter which 3 bits we are given, we are always guaranteed at least two ‘independent’ pieces of information and hence if we cannot determine abc , we can always show that there are only two possible values it could take.

What is remarkable is how this example scales. If we define f_n in a similar manner for all n , then for any string σ of length n , given $1/4$ of the bits of $f_n(\sigma)$ we can find a set of size at most two that σ must belong to. Crucially, the maximum size of the set does not depend on n , but only the fraction of the bits we have access to, in this case $1/4$.

Let V be a vector space of dimension n over a finite field \mathbb{F} of cardinality q . Note that in this paper we are only concerned with the case that $q = 2$, but we will present the argument in its general form. Denote by V^* the dual space of V , i.e., the set of linear functionals from V to \mathbb{F} . In the example above, we were given the values of some bits of $f_n(a, b, c)$. Each bit told us two things, the position of the bit told us the linear functional used, and the value of the bit told us the value of the linear functional on the argument abc . We can represent this information as a subset S of pairs in $(\varphi, z) \in V^* \times \mathbb{F}$. For any $x \in V$, let $C_x = \{(\varphi, \varphi(x)) \in V^* \times \mathbb{F} \mid \varphi \in V^*\}$. The set C_x simply records the values x takes on each linear functional in V^* . Now if $S \subseteq C_x$, then for all $(\varphi, z) \in S$, we have that $\varphi(x) = z$ i.e. the values that x takes on the linear functionals in V^* is consistent with the information provided by S .

Proposition 7. *Let V, n, \mathbb{F}, q, V^* and C_x be defined as above. For any finite set S of pairs $(\varphi, z) \in V^* \times \mathbb{F}$, the set $\mathcal{A}_S = \{x \mid S \subseteq C_x\}$ has cardinality at most $q^n/|S|$.*

Proof. Fix a finite set S of pairs $(\varphi, z) \in V^* \times \mathbb{F}$. If $\mathcal{A}_S = \emptyset$ then the result holds trivially so let us assume that \mathcal{A}_S is not empty and fix some element $x_0 \in \mathcal{A}_S$. If some ψ appears in two different pairs (ψ, z_1) and (ψ, z_2) of S , then it is clear that $\mathcal{A}_S = \emptyset$; so we may assume that any φ appears at most once in the pairs of S . Call the domain of S (written $\text{dom}(S)$) those φ that are the first coordinate of some element of S and let H be the subspace of V^* generated by $\text{dom}(S)$. Since $|H| \geq |S|$ we have $\dim(H) = \log_q(|H|) \geq \log_q(|S|)$, where the equality is due to the linear independence of vectors in a basis.

If $y \in \mathcal{A}_S$, this means that for all $\varphi \in \text{dom}(S)$, $\varphi(x_0) = \varphi(y)$, and therefore by linearity, that $\varphi(x_0) = \varphi(y)$ for all $\varphi \in H$, which again by linearity can be re-written as $\varphi(x_0 - y) = 0$ for all $\varphi \in H$. In other words, $x_0 - y$ belongs to the annihilator H° of H . Of course, the converse

holds, i.e. if $x_0 - y$ belongs to H° , then $\varphi(x_0) = \varphi(y)$ for all $\varphi \in \text{dom}(S)$ and therefore $y \in \mathcal{A}_S$. This shows that \mathcal{A}_S , provided it contains at least one element x_0 , is the affine space $x_0 + H^\circ$ and therefore has the same dimension as H° . Now, using the classical expression of the dimension of the annihilator, $\dim(\mathcal{A}_S) = \dim(H^\circ) = \dim(V^*) - \dim(H) = n - \dim(H) \leq n - \log_q(|S|)$. This implies that $|\mathcal{A}_S| \leq q^n/|S|$. \square

We will also need the following two easy lemmas about density.

Lemma 8. *For all integers n let $I_n = \{2^n, \dots, 2^{n+1} - 1\}$ (note that the I_n form a partition of $\mathbb{N} \setminus \{0\}$). If a set $D \subseteq \mathbb{N}$ has positive upper density greater than δ , then for infinitely many n , the density of D inside I_n (that is, the quantity $|D \cap I_n|/|I_n|$) is at least $\delta/2$.*

Proof. Suppose for the sake of contradiction that $|D \cap I_n|/|I_n| < \delta/2$ for almost all n . Without loss of generality, we may assume that this even holds for all n (by removing finitely many elements from D , which does not change the upper density). For any given k , let $n = n(k)$ be such that $k \in I_n = \{2^n, \dots, 2^{n+1} - 1\}$. Then

$$|D \cap \{0, \dots, k\}| \leq \sum_{j \leq n} |D \cap I_j| \leq \sum_{j \leq n} (\delta/2) \cdot 2^j \leq (\delta/2) \cdot 2^{n+1} \leq (\delta/2) \cdot 2k \leq \delta k.$$

This contradicts the fact that the upper density of D is greater than δ . \square

Lemma 9. *If $D \subseteq \mathbb{N}$ has upper density δ and $E \subseteq \mathbb{N}$ has density 0 then $D \setminus E$ has upper density δ .*

Proof. For all n , $|(D \setminus E) \cap \{0, \dots, n-1\}| \geq |D \cap \{0, \dots, n-1\}| - |E \cap \{0, \dots, n-1\}| \geq |D \cap \{0, \dots, n-1\}| - o(n)$. Thus $\limsup_n |(D \setminus E) \cap \{0, \dots, n-1\}|/n = \limsup_n |D \cap \{0, \dots, n-1\}|/n$ \square

We are now ready to prove Theorem 5.

Proof of Theorem 5. We construct Φ block by block. On input A , for all n , the initial segment $A \upharpoonright n$ is mapped to a string σ_n of length 2^n , and the image of A under Φ is the sequence $B = \sigma_1 \sigma_2 \sigma_3 \dots$. For all n , σ_n is constructed as follows:

- Identify $\{0, 1\}^n$ with the vector space V_n of dimension n over \mathbb{F}_2 , and let x be the element of V_n corresponding to $A \upharpoonright n$.
- Order the elements of V_n^* in some canonical way (say lexicographically, as V_n^* can also be identified with $\{0, 1\}^n$): $V_n^* = \{\varphi_1, \varphi_2, \dots, \varphi_{2^n}\}$.
- Define σ_n to be the string $\varphi_1(x)\varphi_2(x) \dots \varphi_{2^n}(x)$.

Let us show that if $B = \sigma_1 \sigma_2 \sigma_3 \dots$ is not absolutely undecidable, then A is computable. Assuming B is not absolutely undecidable, let f be a partial computable function whose domain D has upper density greater than some rational $\delta > 0$ and such that $f(k) = B(k)$ for all $k \in D$. By Lemma 8, there are infinitely many n such that the density of D in I_n is

at least $\delta/2$. Since D is c.e. and δ rational, the set of such n is c.e. and therefore contains a computable set $\{n_0 < n_1 < n_2 < n_3 < \dots\}$. Note that $B \upharpoonright I_n = \sigma_n$, so by Lemma 8, for all n_i , we can compute a fraction $\delta/2$ of the values of the bits of σ_{n_i} (taking the values of f on elements in I_{n_i}). By the construction of σ_{n_i} , this means that we can compute a subset of $\{(\varphi, \varphi(x)) \mid \varphi \in V_{n_i}^*\}$ (where x is the element of V_{n_i} corresponding to $A \upharpoonright n_i$) of size at least $(\delta/2) \cdot 2^{n_i}$, which, by Proposition 7, allows us to compute a finite set \mathcal{A}_i of size at most $\frac{2^{n_i}}{(\delta/2)2^{n_i}} = 2/\delta$ containing $A \upharpoonright n_i$. Thus, A belongs to the Π_1^0 class

$$\{X \in 2^\omega \mid \forall i (X \upharpoonright n_i) \in \mathcal{A}_i\}.$$

This Π_1^0 class contains at most $2/\delta$ elements as all \mathcal{A}_i have size at most $2/\delta$. Thus all elements of this Π_1^0 class are computable, and therefore A is computable.

It is clear that $A \leq_{tt} \Phi(A)$, as we only need the first $2^{n+1} - 1$ bits of $\Phi(A)$ to recover $A \upharpoonright n$. \square

The proof of Theorem 5 constructs a single functional Φ that uniformly encodes any sequence $A \in 2^\omega$ to Φ^A . On the other hand, the reader might notice that the decoding procedure given for recovering A given Φ^A on a set of positive upper density is not uniform. Indeed, finding a path in a computable tree which only has finitely many paths cannot be done uniformly in general. And even with a known bound on the number of paths, one cannot effectively compute a finite list of sequences which contains all paths. We shall formally prove that this cannot be avoided, not only for the Walsh-Hadamard coding we use, but for any other coding scheme as well. By analogy to the terminology used above, the next theorem can be informally stated as follows: infinitary erasure codes with finite list decoding do not exist.

In this section we will freely identify 2^ω with the powerset of \mathbb{N} . Fix a tt -functional Γ . A Turing functional Ψ correctly decodes X on D , if Ψ can compute X given access to the bits of Γ^X in D , along with D itself, i.e. if $X = \Psi(D \oplus (\Gamma^X \cap D))$. The following theorem shows that for a fixed δ , there does not exist a tt -functional Γ , together with a finite number of functionals Ψ_1, \dots, Ψ_k such that for any X , and any D such that $\rho(D) \geq \delta$, X belongs to the set $\{\Psi_i(D \oplus (\Gamma^X \cap D)) \mid i \leq k\}$.

Theorem 10. *Fix k . Let Γ be a tt -functional, $\Psi_1, \Psi_2, \dots, \Psi_k$ a list of Turing functionals and σ, τ finite strings. There exist computable X and D with $\sigma \preceq X$ and $\tau \preceq D$ such that:*

(i) $\rho(D) \geq 1/3$.

(ii) $X \notin \{\Psi_i(D \oplus (\Gamma^X \cap D)) \mid i \leq k\}$.

Proof. The proof proceeds by induction on k . For $k = 0$, there is nothing to prove. For $k + 1$, let Γ be a tt -functional, $\Psi_1, \Psi_2, \dots, \Psi_{k+1}$ a list of Turing functionals and σ, τ finite strings. Find a finite string $\sigma' \succeq \sigma$ such that $|\Gamma^{\sigma'}| \geq |\tau|$. Let X_1, X_2, X_3 be three distinct computable infinite binary sequences extending σ' and let $Y_i = \Gamma^{X_i}$ for $i \leq 3$. By the discussion of page 4, for all n , there are $1 \leq i < j \leq 3$ such that $Y_i \upharpoonright n$ and $Y_j \upharpoonright n$ coincide on at least one third of their bits. Thus there are $1 \leq i < j \leq 3$ such that $Y_i \upharpoonright n$ and $Y_j \upharpoonright n$ coincide on at least one

third of their bits for infinitely many n . Without loss of generality, assume this holds for the pair (Y_1, Y_2) . The set D of positions n such that $Y_1(n) = Y_2(n)$ has upper density at least $1/3$. Further as Y_1 and Y_2 must agree on the first $|\tau|$ bits, we can adjust D so that $\tau \preceq D$ without affecting the upper density of D or the fact that for all $n \in D$, $Y_1(n) = Y_2(n)$.

If $X_1 \notin \{\Psi_i(D \oplus (Y_1 \cap D)) \mid i \leq k+1\}$, then X_1 and D witness that the theorem holds for $\Gamma, \Psi_1, \Psi_2, \dots, \Psi_{k+1}, \sigma$ and τ . Otherwise, we can assume without loss of generality that $X_1 = \Psi_{k+1}(D \oplus (Y_1 \cap D))$. Observe that there is some m such that $X_1(m) \neq X_2(m)$. Further as $Y_1 \cap D = Y_2 \cap D$, and $\Psi_{k+1}(D \oplus (Y_2 \cap D)) = X_1$, there exists $\hat{\tau} \preceq D$ and $\hat{\sigma} \preceq X_2$ such that $|\hat{\sigma}| \geq m+1$, and

$$\Psi_{k+1}(\hat{\tau} \oplus (\Gamma^{\hat{\sigma}} \cap \hat{\tau})) \succeq X_1 \upharpoonright (m+1).$$

By our induction hypothesis for $\Gamma, \Psi_1, \Psi_2, \dots, \Psi_k, \hat{\sigma}$ and $\hat{\tau}$, there is \hat{X}, \hat{D} such that $\hat{\sigma} \preceq \hat{X}$, $\hat{\tau} \preceq \hat{D}$, and $\hat{X} \notin \{\Psi_i(\hat{D} \oplus (\Gamma^{\hat{X}} \cap \hat{D})) \mid i \leq k\}$. Observe that $\Psi_{k+1}(\hat{D} \oplus (\Gamma^{\hat{X}} \cap \hat{D})) \succeq X_1 \upharpoonright (m+1) \neq \hat{X} \upharpoonright (m+1)$ so in this case \hat{X} and \hat{D} witness that the theorem holds. \square

4 Between bi-immunity and absolute undecidability

Theorem 5 shows that in every non-trivial tt-degree (and thus Turing degree) there is an absolutely undecidable set. In other words, in every such degree there is a set such that we are unable to infinitely often correctly guess a constant percentage of its bits. We show that the statement of the theorem is tight. Indeed, Theorem 11 below shows that there exists a sequence X such that every Y Turing below X is “close” to not being absolutely undecidable.

In the rest of this paper, $(\varphi_e)_e$ and $(\Phi_e)_e$ denote respectively a standard enumeration of partial computable functions from \mathbb{N} to \mathbb{N} and a standard enumeration of partial Turing functionals from 2^ω to 2^ω . We say that a Turing degree d is *computably dominated* if for any function $f \leq_T d$, there is a computable function g such that for all n , $g(n) \geq f(n)$.

Theorem 11. *There is a non-computable set X such that for all $Y \leq_T X$, and any computable function $h \in o(n)$, there exists a partial computable function φ such that:*

1. Y extends φ (i.e., $Y(n) = \varphi(n)$ for all $n \in \text{dom}(\varphi)$).
2. $\kappa_{\text{dom}(\varphi)} \notin o(h)$.

Theorem 11 is proved using forcing with computable perfect trees. A tree $T \subseteq 2^{<\omega}$ is *perfect* if for all $\sigma \in T$ there is some $\tau \succeq \sigma$ such that both $\tau 0$ and $\tau 1$ are in T . Given a tree $T \subseteq 2^{<\omega}$, define $[T]$ to be the set of paths through T , i.e. $[T] = \{X \in 2^\omega \mid \forall n X \upharpoonright n \in T\}$.

We define a partial order \mathbb{P} on the set of computable perfect trees in $2^{<\omega}$ by $T_0 \leq T_1$ if $[T_0] \subseteq [T_1]$. For any computable $A \in 2^\omega$, the set

$$D_A = \{T \in \mathbb{P} \mid (\forall X \in [T])(X \neq A)\}$$

is dense in \mathbb{P} . For every Turing functional Φ_e , the set

$$E_e = \{T \in \mathbb{P} \mid ((\forall X \in [T])(\Phi_e^X \text{ is not total})) \vee ((\forall X \in [T])(\Phi_e^X \text{ is total}))\}$$

is also dense in \mathbb{P} . Hence if $\{T_i\}_{i \in \mathbb{N}}$ is a descending sequence in \mathbb{P} that meets all such dense sets, then any element of $\bigcap_{i \in \mathbb{N}} [T_i]$ is a non-computable set of computably dominated degree. Further such a sequence is computable by \emptyset'' .

Since a Turing reduction to a set of computably dominated degree is always equivalent to a truth-table reduction it is enough to continue working with the latter class of reductions in the remainder of the proof.

In order to prove Theorem 11 we will show that if $h \in o(n)$ is a computable function and Φ is a truth-table functional, then the set

$$\{T \in \mathbb{P} \mid (\exists e)(\forall X \in [T])(\Phi^X \text{ extends } \varphi_e \wedge \kappa_{\text{dom}(\varphi_e)} \notin o(h))\} \quad (1)$$

is dense in \mathbb{P} .

Fix a computable perfect tree T , a truth-table functional Φ and a computable function h such that $h \in o(n)$. For all $\sigma \in T$ we will define two partial computable functions $f_\sigma : 2^{<\omega} \rightarrow T$ and $g_\sigma : \mathbb{N} \rightarrow \{0, 1\}$ such that if f_σ is total then the downward closure of the range of f_σ is a computable perfect subtree of T . Further if f_σ is total then g_σ will witness that the range of f_σ is in (1).

First let $f_\sigma(\lambda) = \sigma$ (where λ is the empty string). Now assume that f_σ has been defined on all strings of length less than or equal to n and let $\{\sigma_1, \dots, \sigma_{2^n}\} = \{f_\sigma(\tau) \mid |\tau| = n\}$. We search the tree T for some $m > n$, a set of nodes $\{\xi_1, \dots, \xi_{2^n}\} \subseteq T$ and $D \subseteq \{0, \dots, m-1\}$ such that:

- (i) $\forall i \in \{1, \dots, 2^n\}, \xi_i \succeq \sigma_i$.
- (ii) $\forall x \in D, \Phi^{\xi_1}(x) = \Phi^{\xi_2}(x) = \dots = \Phi^{\xi_{2^n}}(x)$.
- (iii) $|D| > h(m)$.

If the search succeeds, then for all $x \in D$ we define $g_\sigma(x)$ to be the common value (i.e. $g_\sigma(x) = \Phi^{\xi_1}(x)$). We set $f(\sigma_i 0)$ to be one immediate successor of ξ_i in T and $f(\sigma_i 1)$ to be the other immediate successor of ξ_i in T .

We claim that there exists some $\sigma \in T$ such that the function f_σ is total. To prove this, we will define a condition \star such that if \star holds for σ then f_σ is total. Further if \star does not hold for any σ then f_λ is total (and though we will not need this, in fact f_σ will be total for any $\sigma \in T$).

Condition \star holds for $\sigma \in T$ if there exists $D \subseteq \mathbb{N}$ of positive upper density such that for all $X, Y \in [T] \cap [\sigma]$, the set $\{x \in D \mid \Phi^X(x) \neq \Phi^Y(x)\}$ has density zero.

As this condition is the central new idea in the proof of Theorem 11, we will attempt to explain the underlying intuition. Essentially, the functional Φ can behave in one of two ways with respect to $[T] \cap [\sigma]$. Firstly, for some large set D , any two elements of $\Phi([T] \cap [\sigma])$ agree on “almost all” $n \in D$. Secondly, given any large set D , there are two elements in $\Phi([T] \cap [\sigma])$ which disagree on a significant amount of D . Our approach for refining T differs under these two cases. In the first case, that is, if condition \star holds on σ , the refinement of T

is simple because no matter which paths we take in $[T] \cap [\sigma]$, their Φ -images will always agree on “almost all” of D . This approach is pursued in Lemma 12. For the second case, consider a situation in which we have a finite number of paths $\{X_0, \dots, X_n\}$ in $[T]$ that agree under Φ on a fixed large set D , that is, if $i, j \in \{0, \dots, n\}$, $m \in D$ then $\Phi^{X_i}(m) = \Phi^{X_j}(m)$. If we take two paths in $[T] \cap [\sigma]$ whose Φ -images disagree on a significant amount D , then one of those paths must agree with paths in $\{X_0, \dots, X_n\}$ on a significant amount of D under Φ . In Lemma 13 we will use this idea to build a suitable refinement of T .

Lemma 12. *If condition \star holds for σ then f_σ is total.*

Proof. Let D be a set of positive upper density that witnesses that condition \star holds for σ . Assume that level n of f_σ has been defined and that this level is equal to $\{\sigma_1, \dots, \sigma_{2^n}\}$. For all $i \in \{1, \dots, 2^n\}$, let X_i be the left-most path of T above σ_i .

For $i \in \{2, 3, \dots, 2^n\}$ define $S_i = \{x \in D \mid \Phi^{X_1}(x) \neq \Phi^{X_i}(x)\}$. According to condition \star , for each i , the set S_i has density 0. Thus if we let $\hat{D} = D \setminus (S_2 \cup S_3 \cup \dots \cup S_{2^n})$ we have that \hat{D} is a set of positive upper density by repeated application of Lemma 9. Further for all $x \in \hat{D}$,

$$\Phi^{X_1}(x) = \Phi^{X_2}(x) = \dots = \Phi^{X_{2^n}}(x).$$

As $h(n) \in o(n)$, there exists some $m > n$ such that $\rho_{\hat{D}}(m) > h(m)/m$. For each i , let $\xi_i \in T$ be a sufficiently long initial segment of X_i such that $\Phi^{\xi_i}(x) \downarrow$ for all $x \leq m$. This means that m , $\{\xi_1, \xi_2, \dots, \xi_{2^n}\}$ and $\hat{D} \cap \{0, \dots, m-1\}$ meet the conditions to define the next level of f_σ and hence the search must end successfully at some point. \square

Lemma 13. *If condition \star does not hold for any $\sigma \in T$ then f_λ is total.*

Proof. Assume that the n -th level of f_λ has been defined and that this level is equal to $\{\sigma_1, \dots, \sigma_{2^n}\}$. We argue by induction; let $D_1 = \mathbb{N}$ and let X_1 be the left-most path above σ_1 .

For the induction step assume that for some $i < 2^n$, we have defined D_i and $X_i \succ \sigma_1, \dots, X_i \succ \sigma_i$ such that D_i is a set of positive upper density and for all $x \in D_i$

$$\Phi^{X_1}(x) = \Phi^{X_2}(x) = \dots = \Phi^{X_i}(x).$$

The set D_i has positive upper density; and since condition \star fails above all $\sigma \in T$, it fails in particular above σ_{i+1} . Therefore there exist $Y, Z \in [T] \cap [\sigma_{i+1}]$ such that the set $E = \{x \in D_i \mid \Phi^Y(x) \neq \Phi^Z(x)\}$ does not have density zero i.e. E has positive upper density. Now we can partition E into the sets: $E_1 = \{x \in E \mid \Phi^Y(x) = \Phi^{X_1}(x)\}$ and $E_2 = \{x \in E \mid \Phi^Z(x) = \Phi^{X_1}(x)\}$. At least one of these sets has positive upper density by Lemma 9. If E_1 has positive upper density we let $D_{i+1} = E_1$ and $X_{i+1} = Y$. Otherwise we let $D_{i+1} = E_2$ and $X_{i+1} = Z$.

Hence the set D_{2^n} has positive upper density and for all $x \in D_{2^n}$ we have

$$\Phi^{X_1}(x) = \Phi^{X_2}(x) = \dots = \Phi^{X_{2^n}}(x).$$

As argued in the previous lemma, this is sufficient to show that the construction of f_λ can continue. \square

Proof of Theorem 11. We will show that if Φ is a truth-table functional and h is a computable function such that $h \in o(n)$, then the set (1) is dense in \mathbb{P} . Take any $T \in \mathbb{P}$. Consider the construction of the functions f_σ and g_σ with respect to T , Φ and h . By Lemmas 12 and 13 for some $\sigma \in T$ the function f_σ is total. Let \hat{T} be the downward closure of the range of f_σ . \hat{T} is a computable perfect tree and $\hat{T} \leq T$. Now for all $X \in [\hat{T}]$, Φ^X extends g_σ . Further $\kappa_{\text{dom}(g_\sigma)} \notin o(h)$ because when f_σ is defined on all strings of length n , for some $m > n$ the construction ensures that $\kappa_{(\text{dom}g_\sigma)}(m) > h(m)$.

Let $\{T_i\}_{i \in \mathbb{N}}$ be a descending sequence in \mathbb{P} that meets all of the sets of the form (1) as well as those dense sets that ensure non-computability and being of computably dominated degree. If $X \in \bigcap_i [T_i]$, then X is non-computable. Now if $Y \leq_T X$ then because X is of computably dominated degree $Y = \Phi^X$ for some truth-table functional Φ . Hence for any computable function $h \in o(n)$, there is a partial computable function g such that Y extends g and $\kappa_{\text{dom}(g)} \notin o(h)$. \square

One should notice that in the above proof, while the condition \star is Σ_2^1 , the construction can be carried out using \emptyset'' because it is not necessary to determine if \star holds. All that is required is to find a string σ such that f_σ is total.

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