Denjoy, Demuth and Density

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These slides *(now!)* at http://db.tt/A3WBx0PT

Paradigm

1. In analysis, measure theory etc. there are statements that are shown to hold “almost everywhere”.
2. **Intuition.** A randomly chosen real number would make the statement true.
3. Using concepts from algorithmic randomness we can try to make this intuition more precise.

Derivatives

1 **Definition.** The *slope* of f between two reals a, b is

$$S_f(a,b) = \frac{f(a) - f(b)}{a - b}.$$

2 **Definition.**

$$\overline{\mathrm{D}}f(z) = \limsup_{h \to 0} S_f(z, z+h) \quad \text{and} \quad \underline{\mathrm{D}}f(z) = \liminf_{h \to 0} S_f(z, z+h)$$

(where h ranges over positive and negative numbers)

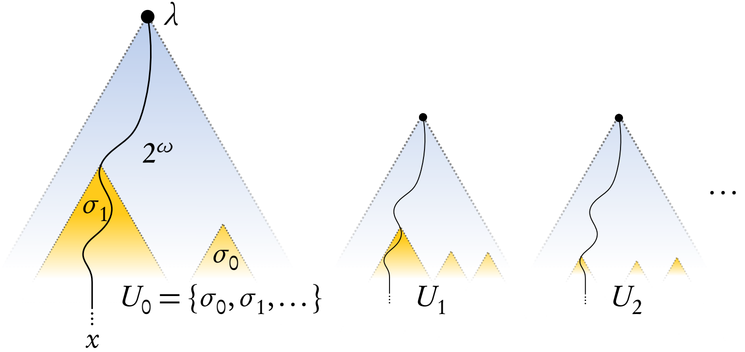
The Denjoy-Young-Saks theorem

1. **Definition.** We say that f satisfies the Denjoy alternative at x if
   - either the derivative of f at x exists
   - or $\overline{D}f(x) = +\infty$ and $\underline{D}f(x) = -\infty$.
2. **Intuition.** Either the derivative exists, or its existence fails in the worst possible way.
3. **The Denjoy-Young-Saks theorem.** For *any* function f and for almost every x, the Denjoy alternative for f at x holds.
4. **Question.** Can we use algorithmic randomness to get a more precise statement than "almost everywhere"?

Algorithmic randomness?

1. **Intuition.** Algorithmic randomness considers a real non-random if it is atypical, i.e., can be covered by a nullset.
2. Useless statement in this form, cf. the nullset $\{x\}$ for any x.
3. **Necessary restriction.** *Effective* nullset.
4. There is a hierarchy of effective notions of nullsets; each giving rise to a measure 1 set of randoms.

Martin-Löf randomness and weak 2-randomness

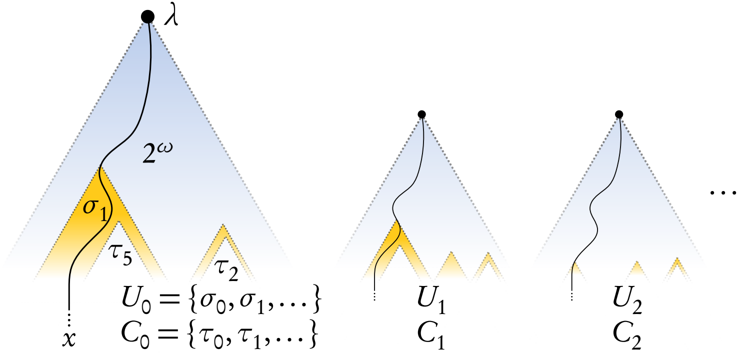


1. **Martin-Löf randomness.** Real is not random if in the intersection of an effective sequence of c.e. classes, whose measure tends to 0 at a guaranteed minimum speed.
2. **Weak 2-randomness.** Ditto, but no speed guarantee.

Martin-Löf randomness and weak 2-randomness

1. **Definition.** A *Martin-Löf test* is a uniformly c.e. sequence $(\mathcal{U}_n)_n$ of open classes such that for all n, $\lambda(\mathcal{U}_n) \leq 2^{-n}$.
2. **Definition.** $x \in 2^\omega$ is called *Martin-Löf random* if for any ML-test $(\mathcal{U}_n)_n$ we have $x \notin \bigcap_n (\mathcal{U}_n)$.
3. **Definition.** A *generalized Martin-Löf test* is a uniformly c.e. sequence $(\mathcal{U}_n)_n$ of open classes such that $\lambda(\mathcal{U}_n) \to 0$.
4. **Definition.** $x \in 2^\omega$ is called *weak 2-random* if for any generalized ML-test $(\mathcal{U}_n)_n$ we have $x \notin \bigcap_n (\mathcal{U}_n)$.

Difference randomness

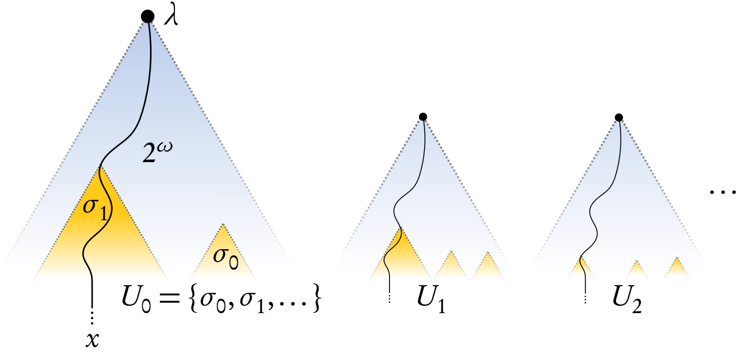


- **Difference randomness.** Like MLR, but we are allowed to fix errors once, by uncovering a covered set again.

Difference randomness

1. **Definition.** A *difference test* is a pair $((\mathscr{U}_n)_n, \mathscr{C})$ of a uniformly c.e. sequence $(\mathscr{U}_n)_n$ of open classes and a single effectively closed class $\mathscr{C}$ such that for all n, $\lambda(\mathscr{U}_n \cap \mathscr{C}) \leq 2^{-n}$.
2. **Definition.** $x \in 2^\omega$ is called *difference random* if for any difference test $((\mathscr{U}_n)_n, \mathscr{C})$ we have $x \notin \bigcap_n (\mathscr{U}_n \cap \mathscr{C})$.

Demuth randomness



- **Demuth randomness.** Here we are allowed to fix errors many times, by resetting the effective covering procedure and restarting from scratch *computably often*.

Demuth randomness

1. **Definition.** A *Demuth test* is a sequence $(\mathcal{U}_n)$ of effectively open sets such that there exists an ω-c.e. function $f : \mathbb{N} \to \mathbb{N}$ which for each n gives a c.e. index for a set of strings generating $\mathcal{U}_n$ and such that for all n, $\lambda(\mathcal{U}_n) \leq 2^{-n}$.
2. **Definition.** $x \in 2^\omega$ is called *Demuth random* if for every Demuth test $(\mathcal{U}_n)$, x belongs to only finitely many $\mathcal{U}_n$.

Computable randomness

1. **Definition.** A martingale is a $d\colon 2^{<\omega} \to [0,\infty)$ such that

$$d(\sigma) = \frac{d(\sigma 0) + d(\sigma 1)}{2} \quad \text{for all } \sigma \in 2^{<\omega}.$$

2. **Definition.** We say $z \in 2^{\omega}$ is *computably random* if and only if for every computable martingale d, $\limsup_n d(z \upharpoonright n) < +\infty$.
3. **Intuition.** No computable martingale wins arbitrary amounts if played against the sequence z.

Hierarchy

1. The following inclusions hold.

$$\begin{array}{ccccccc} \text{W2R} & & & & & & \\ & \searrow & & & & & \\ & & \text{DiffR} & \longrightarrow & \text{MLR} & \longrightarrow & \text{CR} \\ & \nearrow & & & & & \\ \text{DemR} & & & & & & \end{array}$$

2. How much effective randomness is enough to make the Denjoy-Young-Saks theorem true?

Computable real functions

1. In order to be able to work effectively with functions and derivatives we need to make them effective.
2. **Definition.** $f : [0,1] \to \mathbb{R}$ is computable (over the reals) if its value can be effectively approximated with arbitrary precision where the input is provided as an oracle.
3. **Definition.** A real x is computable if it can be effectively approximated with arbitrary precision. Write $x \in \mathbb{R}_c$.
4. **Definition.** $f : \mathbb{R}_c \cap [0,1] \to \mathbb{R}_c$ is said to be Markov computable if from a computable Cauchy name of $x \in \mathbb{R}_c \cap [0,1]$, one can effectively compute a computable Cauchy name for $f(x)$.

Pseudoderivatives

1. Markov computable functions are not necessarily defined on all reals, so $\overline{D}f(z)$ and $\underline{D}f(z)$ may be undefined.
2. Therefore define the following *pseudoderivatives*.
3. **Definition.**

$$\underset{\sim}{D}f(z) = \liminf_{h \to 0^+} \{S_f(a,b) : a, b \in \mathrm{dom}(f) \wedge a \leq x \leq b \wedge 0 < b - a \leq h\}.$$

$$\widetilde{D}f(z) = \limsup_{h \to 0^+} \{S_f(a,b) : a, b \in \mathrm{dom}(f) \wedge a \leq x \leq b \wedge 0 < b - a \leq h\}.$$

(In the case of Markov computable f, $\mathrm{dom}(f)$ is dense in $[0,1]$.)

The effectivized Denjoy alternative

1. **Definition.** Let $f :\subseteq [0,1] \to \mathbb{R}$ be partial with dense domain. f satisfies the effectivized Denjoy alternative at x if
   - either $\widetilde{D}f(x) = \underset{\sim}{D}f(x)$
   - or $\widetilde{D}f(x) = +\infty$ and $\underset{\sim}{D}f(x) = -\infty$.

Known results

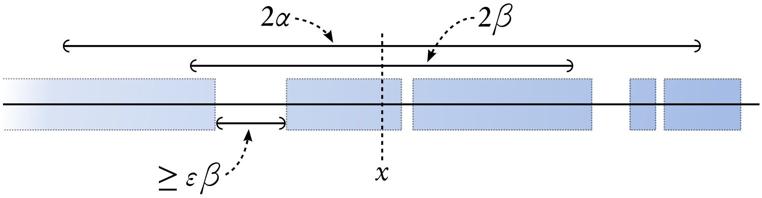
1. **Theorem (Demuth).** Demuth randomness of x is enough to make the effectivized Denjoy alternative at x true for all Markov-computable functions **(notation:** $x \in \mathrm{DA}$).
2. **Theorem (Demuth / Kučera / Brattka, Miller, Nies).** T.f.a.e. for $z \in [0,1]$.
    - z is computably random.
    - For every *computable* f the Denjoy alternative holds at z.

Our first main result

1. **Theorem.** Every difference random real is in DA.
   - **Lemma.** Difference randomness implies *non-porosity*.[1]
   - **Lemma.** Let x be computably random and a non-porosity point. Then $x \in \mathrm{DA}$.
2. This strengthening is significant, since Franklin and Ng showed that DiffR is equivalent to being MLR and having incomplete Turing degree.

[1] defined on next slide

Porosity



1. **Definition.** $\mathscr{C}$ is *porous at x via* $\varepsilon > 0$ if for each $\alpha > 0$ there exists β with $0 < \beta \leq \alpha$ such that $(x - \beta, x + \beta)$ contains an open interval of length $\varepsilon\beta$ that is disjoint from $\mathscr{C}$.
2. **Definition.** $\mathscr{C}$ is *porous at x* if it is porous at x via some ε.
3. **Definition.** x is a *non-porosity point* if every Π^0_1 class to which it belongs is non-porous at x.

Our second main result

1. **Theorem.** DA is incomparable under inclusion with the set of Martin-Löf random reals.
   - **Lemma.** There is a real which is not Martin-Löf random but nonetheless satisfies the Denjoy alternative for all Markov computable functions.
   - **Lemma.** There exists a Markov computable function f for which the Denjoy alternative does not hold at Chaitin's Ω.
2. This is surprising since so far all known randomness notions are.

Lebesgue density

1. **Definition.** Lebesgue lower density of a set $\mathscr{C} \subseteq \mathbb{R}$ at a point $x \in \mathbb{R}$.

$$\rho(x|\mathscr{C}) := \liminf_{\gamma,\delta \to 0^+} \frac{\lambda([x-\gamma, x+\delta] \cap \mathscr{C})}{\lambda([x-\gamma, x+\delta])}$$

2. **Lebesgue density theorem.** Let $\mathscr{C} \subseteq \mathbb{R}$ be measurable. Then $\rho(x|\mathscr{C}) = 1$ for all $x \in \mathscr{C}$ outside a set of measure 0.

A characterization of positive density

1. **Theorem.** T.f.a.e. for $x \in [0,1]$.
   - x is difference random.
   - x is Martin-Löf random and for every Π^0_1 class $\mathscr{C}$ with $x \in \mathscr{C}$ we have $\rho(x|\mathscr{C}) > 0$.
2. Recall that Franklin and Ng showed that this is equivalent to being Martin-Löf random and having incomplete Turing degree.
3. Recently, Day and Miller used this result to prove that K-trivial = ML-non-cuppable.

Density 1 as randomness notion?

1. Again, look at Π^0_1 classes.
2. By definition, every 1-generic is a density 1 point in every Π^0_1 class it is contained in. But they violate basic properties we require from random sequences.
3. So density 1 in Π^0_1 classes does not characterize a randomness notion. For the investigation of this property we therefore only work inside MLR.

Density 1

1. **Proposition.** W2R $\subseteq \{x \mid \forall \Pi^0_1\, \mathscr{C} : x \in \mathscr{C} \Rightarrow \rho(x|\mathscr{C}) = 1\}$.
   Proof. Fix $\mathscr{C}$ and $\varepsilon \in \mathbb{Q}$. Then $\rho(\mathscr{C}|z) < 1 - \varepsilon$ if and only if

   $$\forall \beta > 0\, \exists \gamma, \delta < \beta : \frac{\lambda([z-\gamma, z+\delta] \cap \mathscr{C})}{\gamma + \delta} < 1 - \varepsilon,$$

   which is a Π^0_2 condition.
   The LDT implies that $\{z \in \mathscr{C} \mid \rho(\mathscr{C}|z) < 1 - \varepsilon\}$ is null. □
2. **Open question.** Is W2R equal to this set?

Density and domination

1. **Lemma.** Let $z \in \text{MLR}$ not be a density 1 point. Then z computes a single f dominating every $g \leq_T A$ for those A with $z \in \text{MLR}^A$.
   In particular, such a z is high.
2. **Corollary (Bienvenu, Greenberg, Kučera, Nies, Turetsky).** If $z \in \text{MLR}$ is not a density 1 point, then z computes all K-trivials.
3. **Corollary (Bienvenu, Greenberg, Kučera, Nies, Turetsky; Day, Miller).** There is a single, incomplete MLR set computing all K-trivials.

Density and domination

1. **Definition.** z is *uniformly almost everywhere dominating* if it computes an f dominating every $g \leq_T A$ for almost all A.
2. **Theorem.** Let $z \in$ MLR not be a density 1 point. Then z is uniformly almost everywhere dominating.
3. **Theorem (Kjos-Hanssen, Miller and Solomon).** A is u.a.e.d. iff it is LR-hard.

Conclusion: the picture inside MLR

z is not LR-hard $\longrightarrow$ z is a density 1 point

$\downarrow \not\uparrow$ Day, Miller

z is a positive density point $\longleftrightarrow$ $z \not\geq_T \emptyset'$

$\downarrow$

z is a non-porosity point

Thank you for your attention.