Randomness for computable measures, and complexity

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1 Motivation

Motivation

1. **Theorem (Levin, Schnorr).** $X \in 2^\omega$ is Martin-Löf random iff

$$\forall n\, \mathrm{K}(X \upharpoonright n) \geq n - O(1).$$

2. This version for Lebesgue measure can also be formulated for arbitrary computable measures μ:

   Theorem (Levin, Schnorr). $X \in 2^\omega$ is μ-Martin-Löf random iff

$$\forall n\, \mathrm{K}(X \upharpoonright n) \geq -\log(\mu(X \upharpoonright n)) - O(1).$$

3. **Therefore:** The possible growth rates of K for μ-random sequences are related to the structure of μ.

Goals

1. Study how properties of μ are reflected in the growth rates of K for μ-random sequences.
2. Study the growth rates of K for *proper* sequences, i.e., sequences random for *some* computable measure μ.
3. Study computable measures whose set of randoms is "small."
   (in a sense to be explained)

2 Preliminaries

Measures and atoms

1. **Definition.** μ is *computable* if $\sigma \mapsto \mu(\llbracket \sigma \rrbracket)$ is a computable real-valued function.
2. **Definition.** μ is *atomic* if there is $X \in 2^\omega$ with $\mu(\{X\}) > 0$.
   - Then X is called an *atom* of μ.
   - Atoms_μ is the set of all atoms of μ.
3. **Fact.** Atoms of a computable measure μ are trivially μ-random and computable.
4. **Definition.** If μ is not atomic, then it is *continuous*.

3 Complexity and properness

Complex sequences

1. **Definition.** X is *complex* if there is a computable order $h : \omega \to \omega$ such that

$$\forall n \, \mathrm{K}(X \upharpoonright n) \geq h(n).$$

2. **Intuition.** For complex sequences a certain Kolmogorov complexity growth rate is guaranteed everywhere.

Complexity and properness

1. **Theorem (essentially Bienvenu, Porter).**
   If $X \in 2^\omega$ is μ-Martin-Löf random for μ computable and continuous, then X is complex.
2. The converse is false, as there are complex non-proper sequences.
   - Miller showed that there is a sequence of effective Hausdorff dimension $1/2$ that does not compute a sequence of higher effective Hausdorff dimension.
   - Such a sequence is clearly complex.
   - If it computed any proper sequence, then it would compute an MLR sequence (Zvonkin, Levin), contradiction.

Complexity and properness

1. However, there is a restricted converse for proper sequences.
2. **Theorem (Hölzl, Porter).** Let $X \in 2^\omega$ be proper. If X is complex, then $X \in \mathrm{MLR}_\mu$ for some computable, continuous measure μ.
3. **Proof idea.** The complexity of X allows "patching" the measure to remove the (non-complex) atoms.

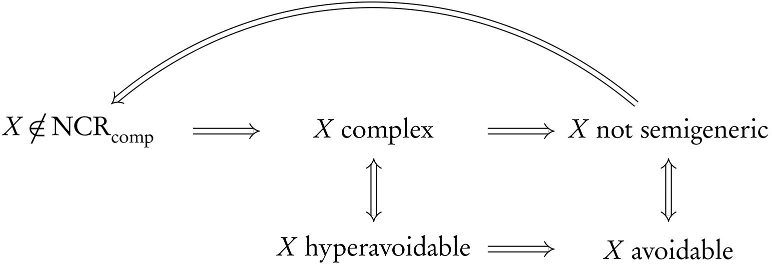
Semigenericity, avoidability,...

1. **Definition (Bienvenu, Porter).** $\mathrm{NCR}_{\mathrm{comp}}$ is the collection of sequences that are not random with respect to any computable, continuous measure.
2. **Definition (Demuth).** $X \in 2^\omega$ is *semigeneric* if for every Π^0_1 class $\mathscr{P}$ with $X \in \mathscr{P}$, $\mathscr{P}$ contains a computable member.
3. **Definition (Miller).** $X \in 2^\omega$ is *avoidable* if there is a partial computable function p such that for every computable set M and every c.e. index e for M, we have $p(e)\downarrow$ and $X \upharpoonright p(e) \neq M \upharpoonright p(e)$.
4. **Definition (Miller).** X is *hyperavoidable* if X is avoidable with total p.

... and randomness

$$
\begin{array}{ccccc}
X \notin \mathrm{NCR}_{\mathrm{comp}} & \underset{\text{ex.}}{\overset{\text{B, P}}{\substack{\Longrightarrow \\ \not\Longleftarrow}}} & X \text{ complex} & \substack{\Longrightarrow \\ \not\Longleftarrow} & X \text{ not semigeneric} \\
 & & \Updownarrow \text{KH, M, S} & & \Updownarrow \text{D, K} \\
 & & X \text{ hyperavoidable} & \underset{\text{M}}{\substack{\Longrightarrow \\ \not\Longleftarrow}} & X \text{ avoidable}
\end{array}
$$

Theorem (Hölzl, Porter). Let $X \in 2^\omega$ be proper, non-computable.



... and randomness

Theorem (Hölzl, Porter). Let $X \in 2^\omega$ be proper, non-computable.

$$X \notin \mathrm{NCR}_{\mathrm{comp}} \iff X \text{ complex} \iff X \text{ not semigeneric}$$

$$X \text{ complex} \Updownarrow X \text{ hyperavoidable}, \qquad X \text{ not semigeneric} \Updownarrow X \text{ avoidable}$$

$$X \text{ hyperavoidable} \iff X \text{ avoidable}$$

Complexity in a uniform sense?

1. **Question.** For given computable and continuous μ, is there a single computable order function witnessing complexity of μ-random sequences?

Granularity

1. **Definition (Reimann, Slaman).** For μ continuous, the *granularity of* μ is defined as

$$g_\mu\colon\ n \mapsto \min\{\ell\colon\ \forall \sigma \in 2^\ell : \mu(\llbracket \sigma \rrbracket) < 2^{-n}\}.$$

2. **Theorem (Hölzl, Porter).** If μ is continuous and computable, there is a computable order h such that $|h(n) - g_\mu^{-1}(n)| \leq O(1)$ and for every $X \in \mathrm{MLR}_\mu$, $\mathrm{K}(X \upharpoonright n) \geq h(n)$.
3. g_μ^{-1} provides a global lower bound for the initial segment complexity of every μ-random sequence.
4. g_μ itself is in general not computable, but g_μ^{-1} can be replaced by the computable h above.

4 Atomic measures

Removability of atoms

1. **Question.** If we have a computable, atomic measure μ such that

$$\forall X \in 2^{\omega}\ (X \in \mathrm{MLR}_{\mu} \setminus \mathrm{Atoms}_{\mu} \Rightarrow X \text{ is complex}),$$

is there a computable, continuous measure ν such that

$$\mathrm{MLR}_{\mu} \setminus \mathrm{Atoms}_{\mu} \subseteq \mathrm{MLR}_{\nu}?$$

2. **Theorem (Hölzl, Porter).** There is a computable, atomic measure μ such that
   - every $X \in \mathrm{MLR}_{\mu} \setminus \mathrm{Atoms}_{\mu}$ is complex but
   - there is no computable, continuous measure ν such that $\mathrm{MLR}_{\mu} \setminus \mathrm{Atoms}_{\mu} \subseteq \mathrm{MLR}_{\nu}$.
3. **Intuition.** There are measures with non-removable atoms.

Contrast between continuous and atomic measures

1. During the proof of the previous theorem we established that unlike for continuous measures, **for atomic measures there is no "complexity in a uniform sense."**

   That is, there is in general no uniform computable lower bound for the K-growth rates of the non-atom μ-random sequences.

Contrast between continuous and atomic measures

1. **Theorem, restated.**
   If $X \in 2^\omega$ is μ-Martin-Löf random for μ computable and continuous, then X is complex.
2. **Theorem (Hölzl, Porter).** Every hyperimmune random Turing degree contains a proper sequence that is both i.o. complex and i.o. anti-complex.

   The proof involves the construction of an atomic measure.

5 Trivial and diminutive measures

Trivial and diminutive measures

1. **Definition.** μ is *trivial* if $\mu(\text{Atoms}_\mu) = 1$.
2. **Definition.**
   - **(Binns)** $\mathscr{C} \subseteq 2^\omega$ is *diminutive* if it does not contain a computably perfect subclass.
   - **(Porter)** Let μ be a computable measure, and let $(\mathscr{U}_i)_{i\in\omega}$ be the universal μ-Martin-Löf test. Then we say that μ is *diminutive* if $\mathscr{U}_i^c$ is a diminutive Π^0_1 class for every i.
3. **Intuition.** The collection of randoms is "small" for both types of measures.
   - The randoms for a trivial measure may be of two types:

     countably many atoms   measure 0 many non-atoms
   - The set of randoms for a diminutive measure has strong effective measure 0 (Higuchi, Kihara).

Diminutive measures are more general than trivial ones

1. **Proposition (Hölzl, Porter).** Every computable trivial measure is diminutive.
2. **Theorem (Hölzl, Porter).** There is a computable diminutive measure that is not trivial.
3. As a corollary to the proof, we obtain a priority-free proof of the following known result.

   Corollary (Kautz). There is a computable, non-trivial measure μ such that there is no Δ^0_2, non-computable $X \in \mathrm{MLR}_\mu$.

Thank you for your attention.

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