Randomness for computable measures, and complexity

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1 Motivation

Motivation

1. **Theorem (Levin, Schnorr).** $X \in 2^\omega$ is Martin-Löf random iff

$$\forall n\, \mathrm{K}(X \upharpoonright n) \geq n - O(1).$$

2. This is the special case for Lebesgue measure λ of this general statement for arbitrary computable measures μ:

   Theorem (Levin, Schnorr). $X \in 2^\omega$ is μ-Martin-Löf random iff

$$\forall n\, \mathrm{K}(X \upharpoonright n) \geq -\log(\mu([\![X \upharpoonright n]\!])) - O(1).$$

   (For completeness, let $\log(0) = -\infty$ by convention.)

3. **Therefore:** The possible growth rates of K for μ-random sequences are related to the structure of μ.

Goals

1. Study how properties of μ are reflected in the growth rates of K for μ-random sequences.
2. Study the growth rates of K for *proper* sequences, that is, sequences random for *some* computable measure μ.
3. Use the techniques and results to study computable measures whose set of randoms is "small" (in a sense to be explained).

2 Preliminaries

Measures and atoms

1. **Definition.** μ is *computable* if $\sigma \mapsto \mu(\llbracket \sigma \rrbracket)$ is a computable real-valued function.
2. **Definition.** μ is *atomic* if there is $X \in 2^\omega$ with $\mu(\{X\}) > 0$.
   - Then X is called an *atom* of μ.
   - Atoms_μ is the set of all atoms of μ.
3. **Fact.** Atoms of a computable measure μ are trivially μ-random and computable.
4. **Definition.** If μ is not atomic, then it is called *continuous*.

3 Properness, atoms, complexity

Complex sequences

1. **Definition.** X is *complex* if there is a computable order $h:\omega\to\omega$ such that

$$\forall n\, \mathrm{K}(X\upharpoonright n)\geq h(n).$$

2. **Intuition.** For complex sequences a certain Kolmogorov complexity growth rate is guaranteed everywhere.

From continuity to complexity

1. **Theorem (Bienvenu, Porter).**
   If $X \in 2^\omega$ is μ-Martin-Löf random for μ computable and continuous, then X is complex.
2. The converse is false, as there are complex non-proper sequences.
   - Miller showed that there is a sequence of effective Hausdorff dimension 1/2 that does not compute a sequence of higher effective Hausdorff dimension.
   - Such a sequence is clearly complex.
   - If it computed any (non-computable) proper sequence, then it would compute an MLR sequence **(Zvonkin, Levin; Kautz)**, contradiction.
3. **Question.** For given computable and continuous μ, is there a *single* computable order function witnessing complexity of μ-random sequences?

From complexity to continuity

1. There is a restricted converse of the Theorem.
2. **Theorem (Hölzl, Porter).** Let $X \in 2^\omega$ be proper. If X is complex, then $X \in \mathrm{MLR}_\mu$ for some computable continuous measure μ.
3. **Proof idea.**
   - Let ν be a computable non-continuous measure witnessing X's properness.
   - The complexity of X allows "patching" ν to remove the (non-complex) atoms without affecting X's randomness. $\square$
4. The proof uses a fixed f witnessing the complexity of X.

Question. Can we remove the atoms, while protecting the randomness of *all* non-atom random sequences?

Granularity

1. **Definition (Reimann, Slaman).** For μ continuous, the *granularity of* μ is defined as

$$g_\mu\colon\ n \mapsto \min\{\ell\colon\ \forall \sigma \in 2^\ell\colon \mu(\llbracket \sigma \rrbracket) < 2^{-n}\}.$$

2. **Lemma (Hölzl, Porter).** If μ is continuous and computable, there is a computable order h such that $|h(n) - g_\mu^{-1}(n)| \leq O(1)$ and for every $X \in \mathrm{MLR}_\mu$, $\mathrm{K}(X \upharpoonright n) \geq h(n)$.

3. **Intuition.**
   - g_μ^{-1} provides a global lower bound for the initial segment complexity of *every* μ-random sequence.
   - g_μ itself is in general not computable, but g_μ^{-1} can be replaced by the computable h above.

Nonremovability of atoms

1. **Question, restated.** For a computable, atomic measure μ with

$$\forall X \in 2^\omega\ (X \in \mathrm{MLR}_\mu \setminus \mathrm{Atoms}_\mu \Rightarrow X \text{ is complex}),$$

is there a computable, continuous measure ν such that

$$\mathrm{MLR}_\mu \setminus \mathrm{Atoms}_\mu \subseteq \mathrm{MLR}_\nu?$$

2. **Theorem (Hölzl, Porter).** No. For some μ, there is no such ν.

Nonremovability of atoms

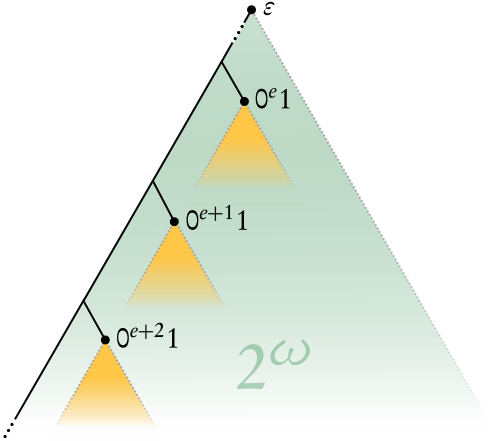
Proof sketch.

1. Atomic measures obviously have no granularity function.
2. **Definition.** But we can define a *local granularity function*

$$g_\mu^X(n) = \min\{\ell : \mu([\![X \upharpoonright \ell]\!]) < 2^{-n}\}.$$

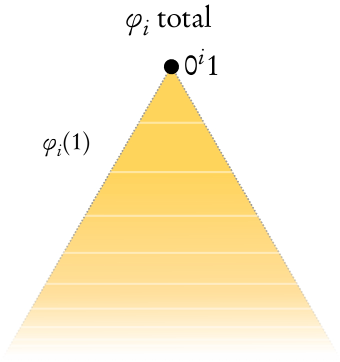
3. Suppose there is a computable, continuous measure ν such that $\mathrm{MLR}_\mu \setminus \mathrm{Atoms}_\mu \subseteq \mathrm{MLR}_\nu$.
4. By the Lemma there is a common computable order h witnessing the complexity of all $X \in \mathrm{MLR}_\nu \supseteq \mathrm{MLR}_\mu \setminus \mathrm{Atoms}_\mu$.
5. One can show that then $g_\mu^X(n)$ for all such X is dominated by (a slight modification of) this single h.
6. So to obtain a contradiction, we need to build a μ such that for *every* computable order h there is an $X \in \mathrm{MLR}_\mu \setminus \mathrm{Atoms}_\mu$ for which g_μ^X dominates h.

Nonremovability of atoms



Cone $[\![0^e1]\!]$ is used to defeat φ_e, if it is a computable order.
If φ_e is partial we ensure that all randoms in $[\![0^e1]\!]$ are atoms.

Nonremovability of atoms



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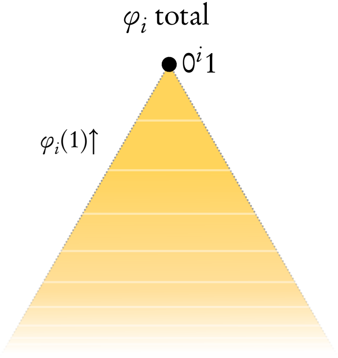

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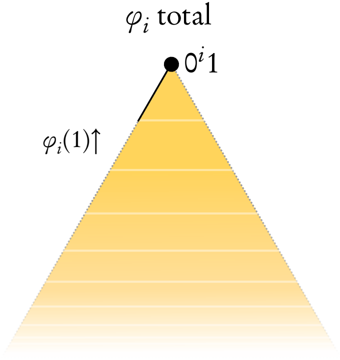

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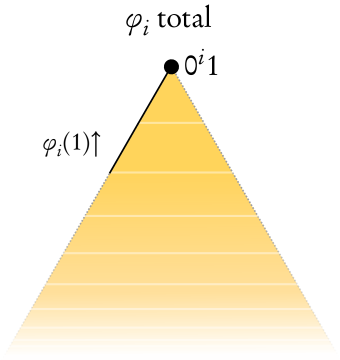

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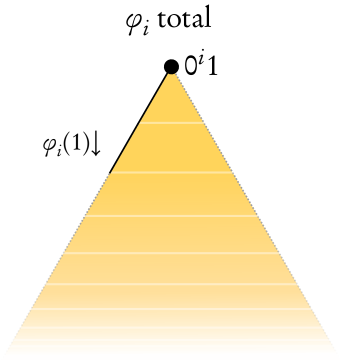

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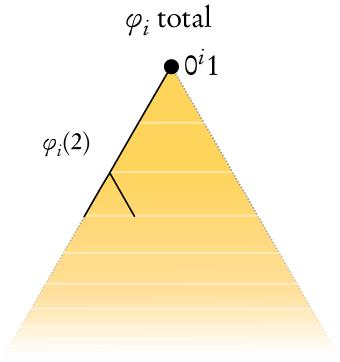

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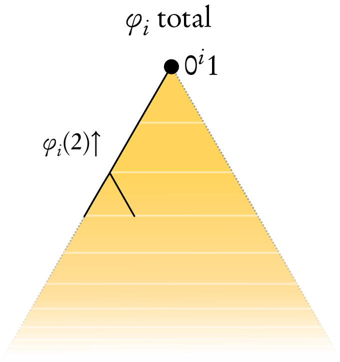

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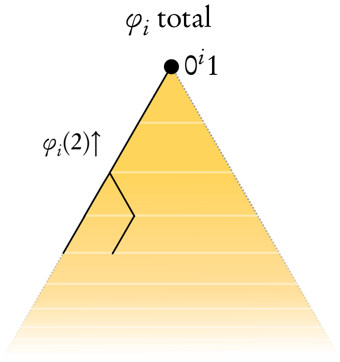

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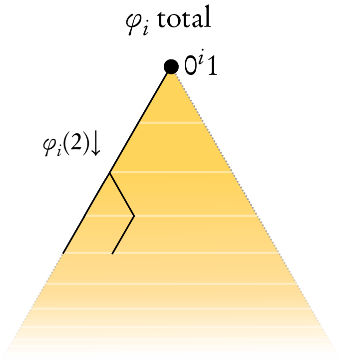

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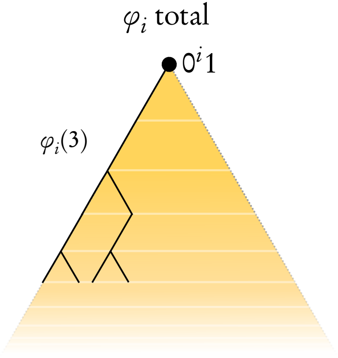

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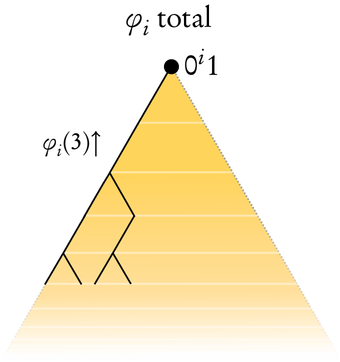

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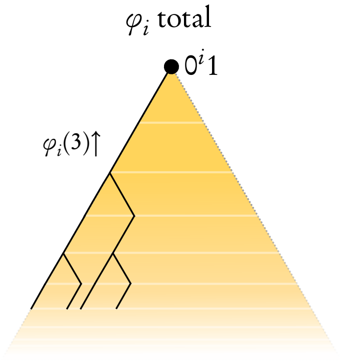

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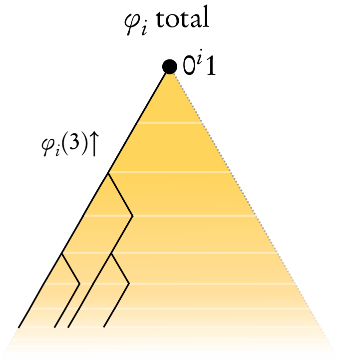

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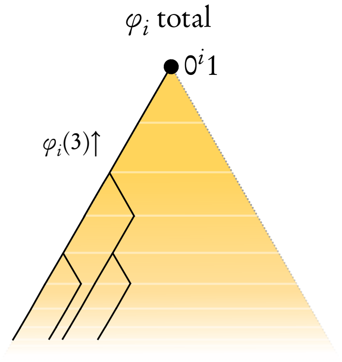

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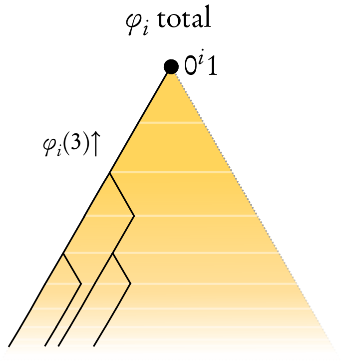

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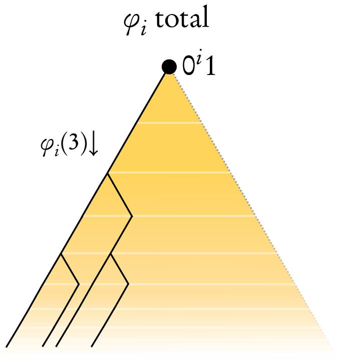

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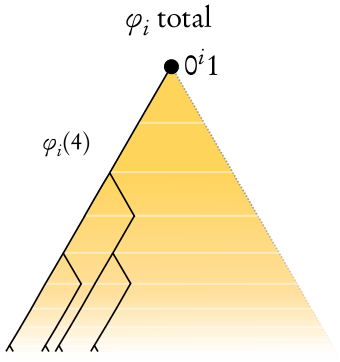

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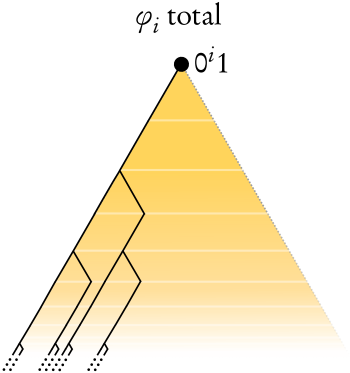

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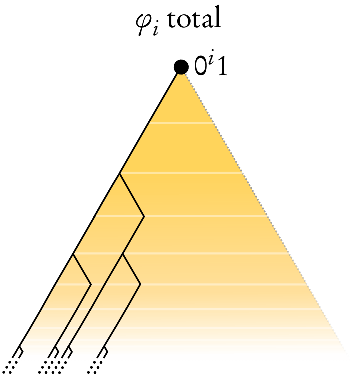
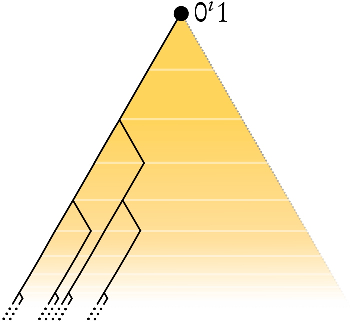


sparse, infinite splitting

Nonremovability of atoms

φ_i total



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$\Downarrow$

g_μ^X dominates φ_i for

$X \in \mathrm{MLR}_\mu \cap [\![0^i1]\!]$

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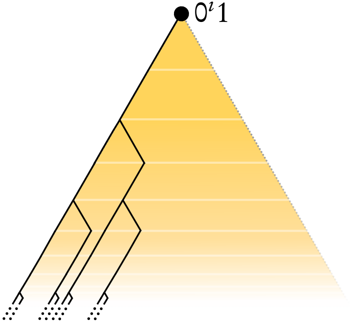

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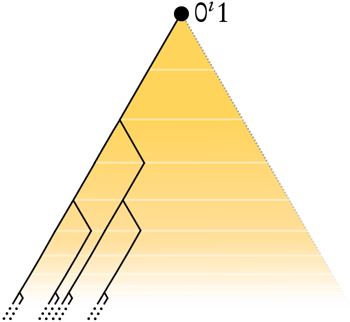

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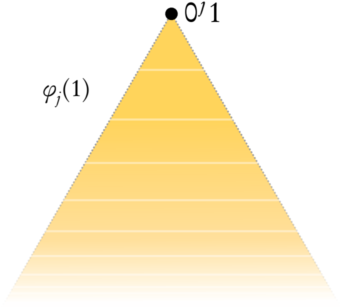

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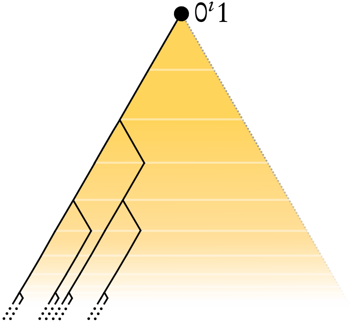

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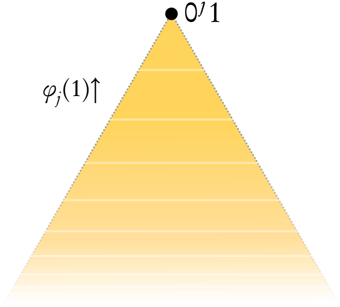

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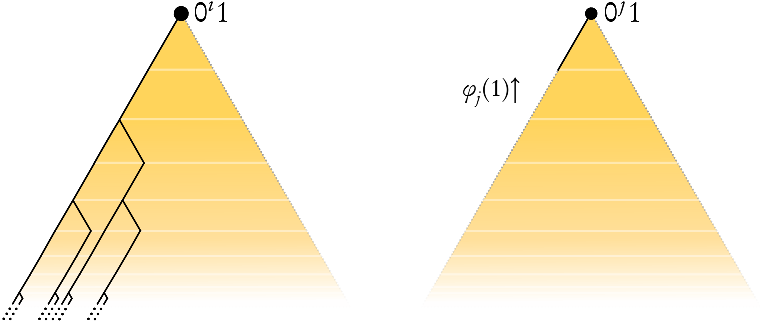

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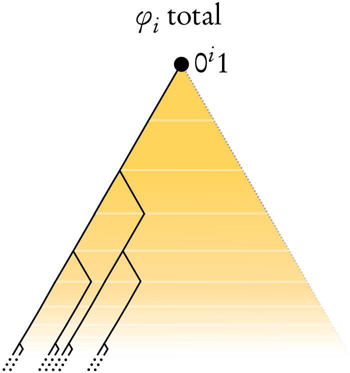

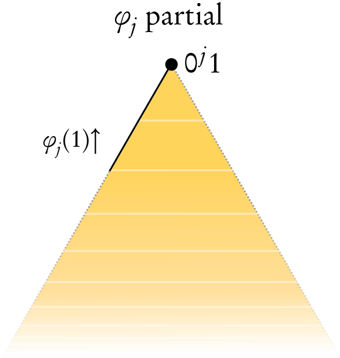

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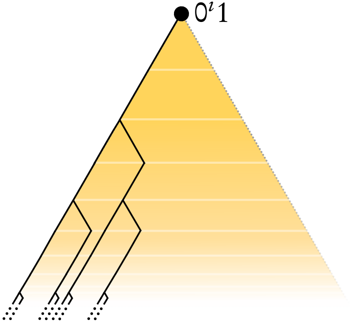

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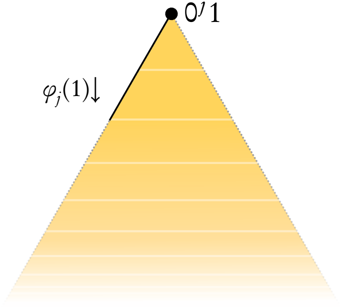

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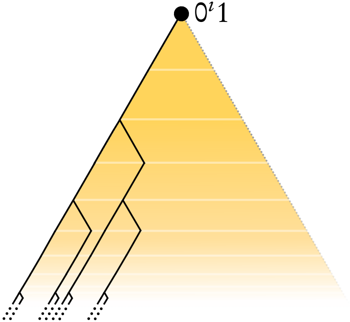

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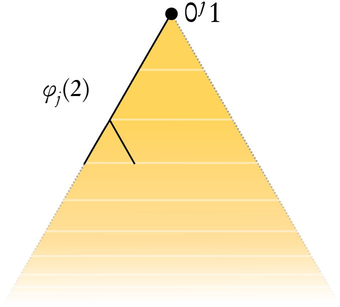

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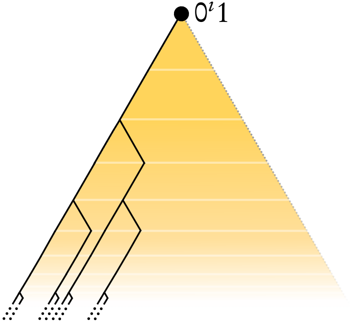

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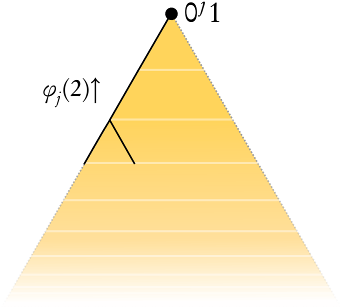

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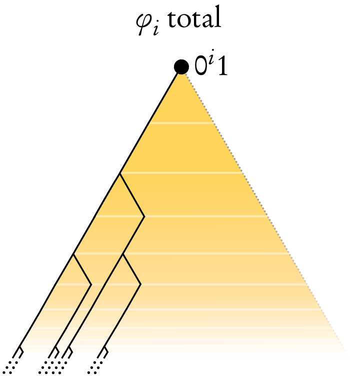

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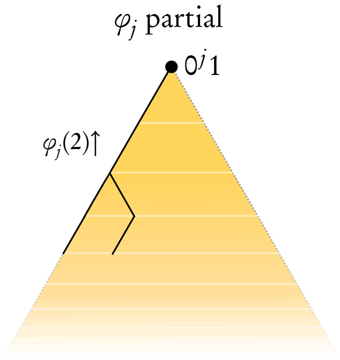

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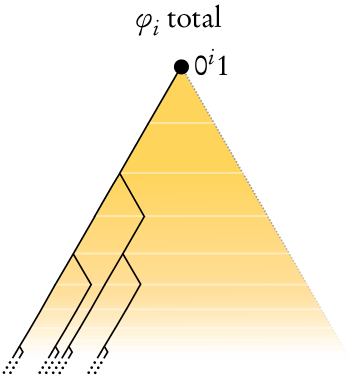

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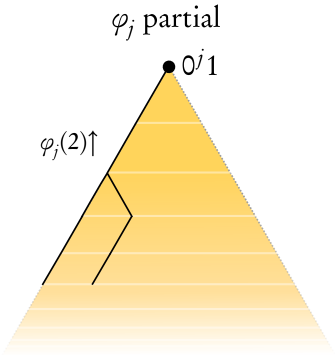

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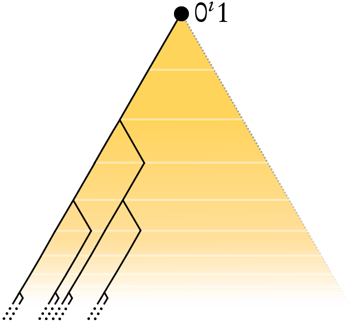

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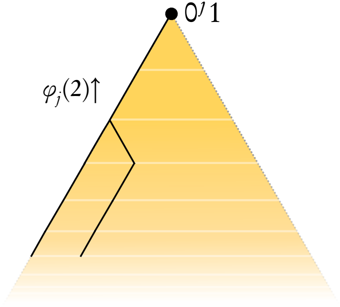

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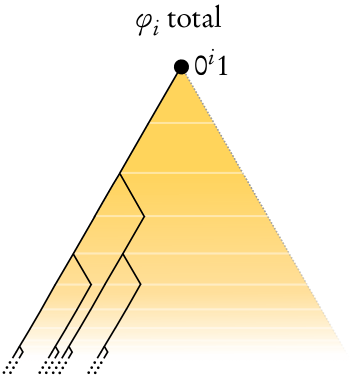

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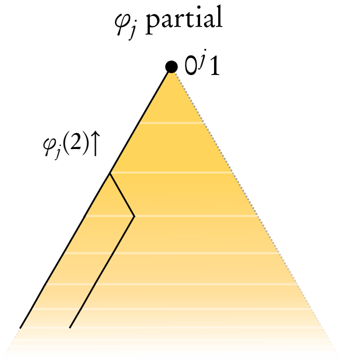

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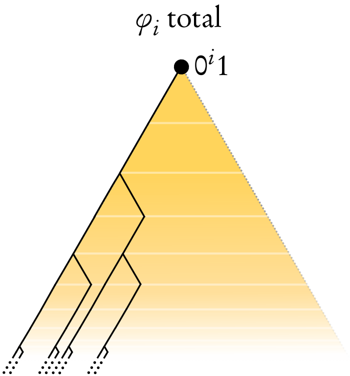

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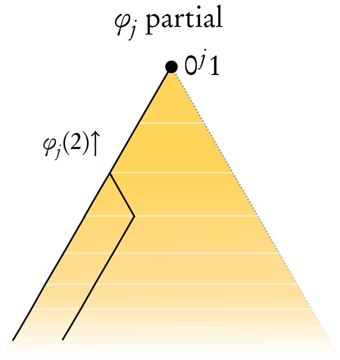

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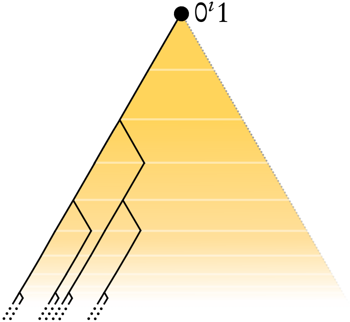

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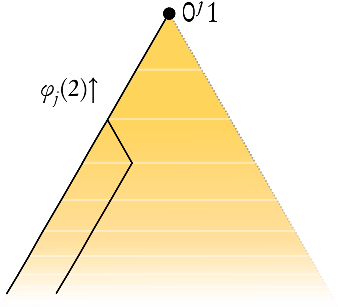

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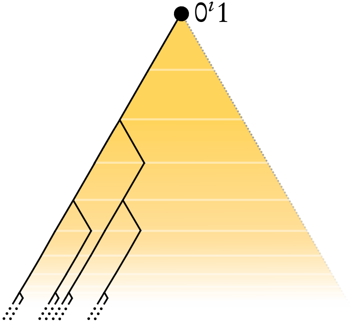

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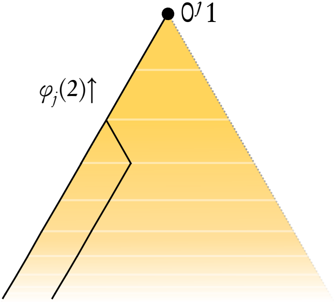

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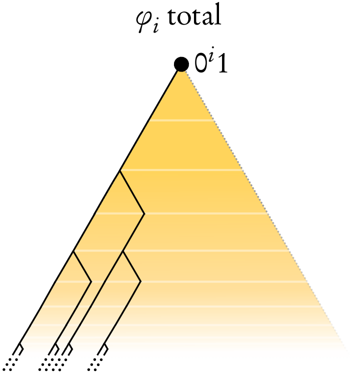

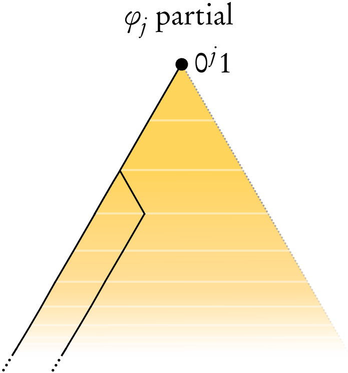

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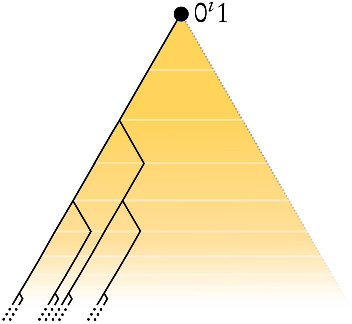
Nonremovability of atoms

φ_i total



sparse, infinite splitting

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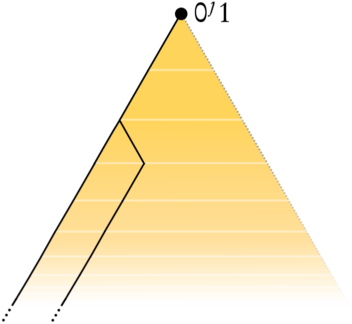

finitely many splits

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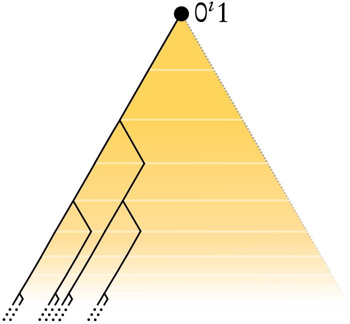

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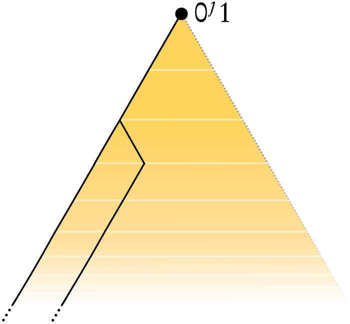

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all randoms are atoms in $[\![0^j1]\!]$

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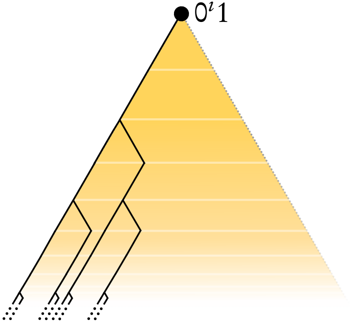

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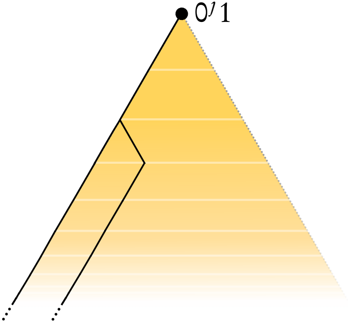


finitely many splits

$\Downarrow$

all randoms are atoms in $[\![0^j 1]\!]$

4 Trivial and diminutive measures

Trivial and diminutive measures

1. **Definition.** μ is *trivial* if $\mu(\text{Atoms}_\mu) = 1$.
2. **Definition.**
   - **(Binns)** $\mathscr{C} \subseteq 2^\omega$ is *diminutive* if it does not contain a computably perfect subclass.
   - **(Porter)** Let μ be a computable measure, and let $(\mathscr{U}_i)_{i\in\omega}$ be the universal μ-Martin-Löf test. Then μ is *diminutive* if $\mathscr{U}_i^c$ is diminutive for every i.
3. **Intuition.** The collection of randoms is "small" for both types of measures.
   - **(Higuchi, Kihara)** The set of randoms for a diminutive measure has strong effective measure 0.
   - The randoms for a trivial measure may be of two types:

     countably many atoms   measure 0 many non-atoms

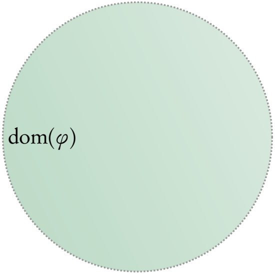
A non-trivial diminutive measure

1. **Proposition (Hölzl, Porter; based on Binns).**
   A computable measure μ is diminutive if and only if there is no complex $X \in \mathrm{MLR}_\mu$.
2. **Proposition (Hölzl, Porter).** Every computable trivial measure is diminutive.
3. **Theorem (Hölzl, Porter).** There is a computable diminutive measure μ that is not trivial.
4. **Proof idea.** Build a μ that is non-zero only on non-complex sequences, while maintaining $\mu(\mathrm{Atoms}_\mu) < 1$.

A non-trivial diminutive measure

1. **(Kautz)** There is a φ with $\lambda(\mathrm{dom}(\varphi)) > 0$ and, for $X \in \mathrm{dom}(\varphi)$, φ^X is not dominated by a computable function; $\mathrm{dom}(\varphi) \in \Pi^0_2$.

A non-trivial diminutive measure



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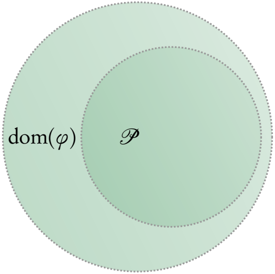

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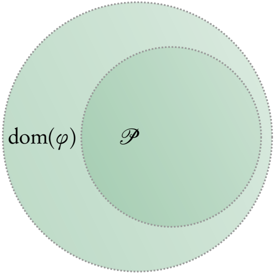
2. **(Kautz)** There are j and $\mathrm{dom}(\varphi) \supseteq \mathscr{P} \in \Pi^{0,\emptyset'}_1$ with $\mu(\mathscr{P}) > 2^{-j}$.

A non-trivial diminutive measure



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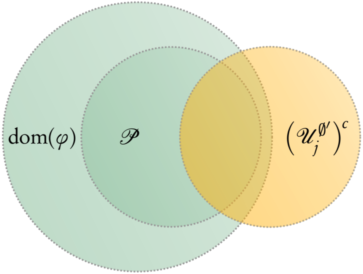
A non-trivial diminutive measure



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3. Let $(\mathscr{U}_i^{\emptyset'})_{i \in \omega}$ be the universal $\emptyset'$-Martin-Löf test.
4. So $\left(\mathscr{U}_j^{\emptyset'}\right)^c \in \Pi_1^{0,\emptyset'}$ and $\lambda\left(\left(\mathscr{U}_j^{\emptyset'}\right)^c\right) > 1 - 2^{-j}$.

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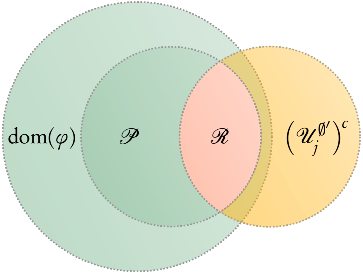

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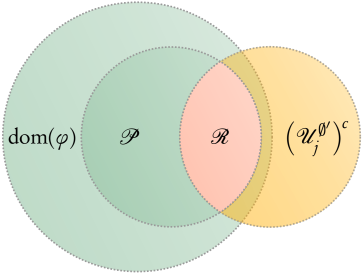
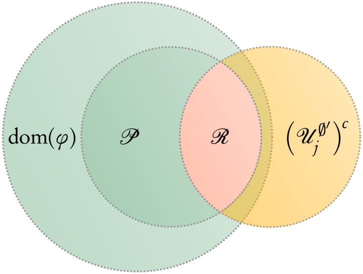
5. Let $\mathscr{R} = \mathscr{P} \cap \left(\mathscr{U}_j^{\emptyset'}\right)^c$.

A non-trivial diminutive measure



6 Then we have

A non-trivial diminutive measure



6 Then we have
 - φ^X is total for all $X \in \mathscr{R}$,

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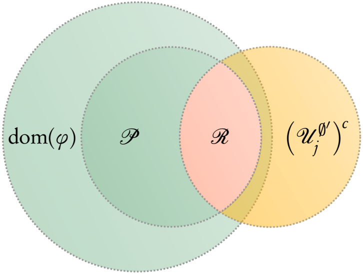

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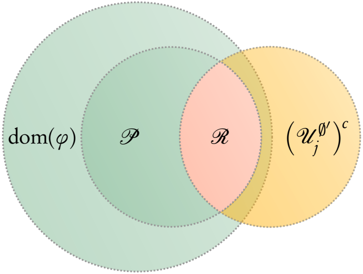

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- φ^X is total for all $X \in \mathscr{R}$,
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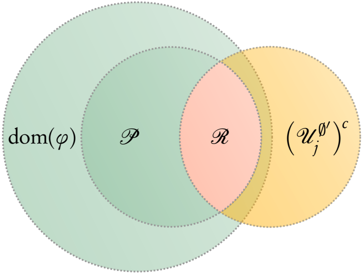
A non-trivial diminutive measure



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- for all $X \in \mathscr{R}$, φ^X is not dominated by a computable function,
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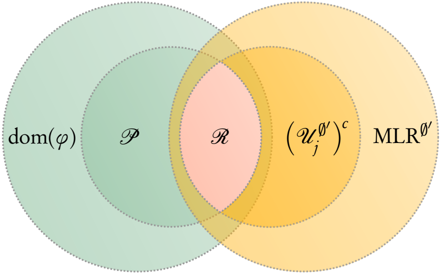

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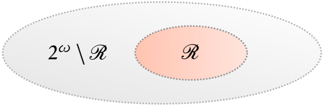
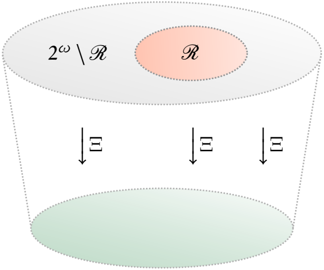
- $\mathscr{R} \subseteq \mathrm{MLR}^{\emptyset'}$.

A non-trivial diminutive measure



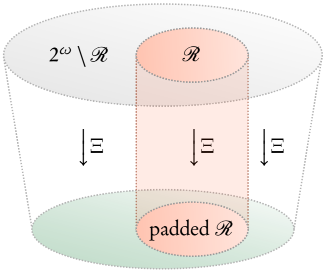
1. We will apply two functionals to $\mathscr{R}$.

A non-trivial diminutive measure



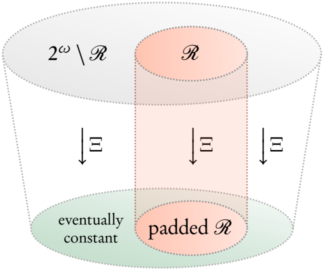
1. We will apply two functionals to $\mathscr{R}$.
2. The first functional Ξ
   (inspired by a construction of Ng, Stephan, Yang, Yu)

A non-trivial diminutive measure



1. We will apply two functionals to $\mathscr{R}$.
2. The first functional Ξ
   (inspired by a construction of Ng, Stephan, Yang, Yu)
   - uses a computable approximation to T to try to find longer and longer initial segments of the input in it;
   - whenever progress is made, outputs one more bit of the input;
   - while waiting for progress, outputs padding bits;
   - thus, maps all $X \in \mathscr{R}$ to Turing-equivalent heavily padded versions;

A non-trivial diminutive measure



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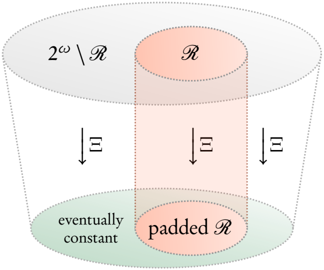

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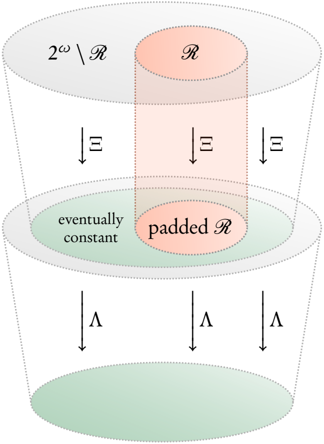
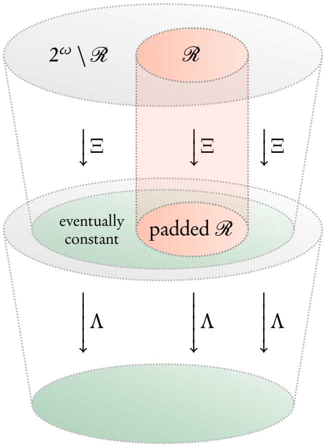
3. This makes Ξ total.

A non-trivial diminutive measure



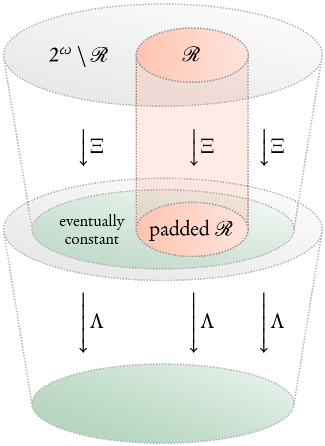
1. The second functional Λ

A non-trivial diminutive measure



1. The second functional Λ
   - tries to find more and more coded bits in the padded sequences in $\Xi(\mathscr{R})$;

A non-trivial diminutive measure



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   - tries to find more and more coded bits in the padded sequences in $\Xi(\mathscr{R})$;
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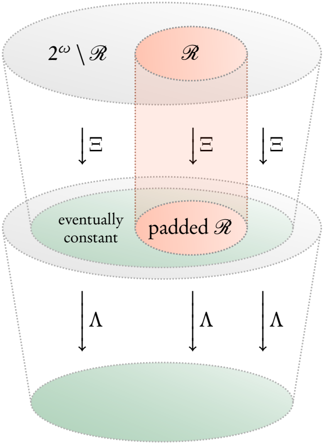
A non-trivial diminutive measure



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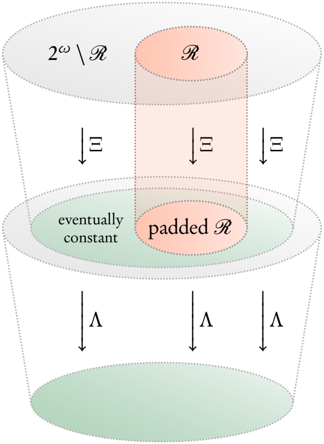
A non-trivial diminutive measure



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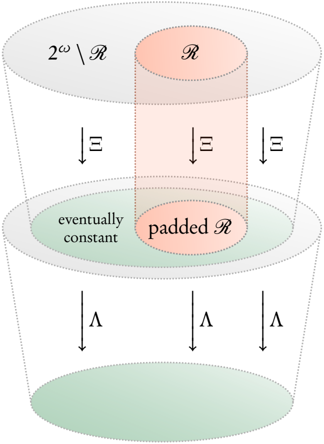
2. For inputs in $\Xi(\mathscr{R})$,

A non-trivial diminutive measure



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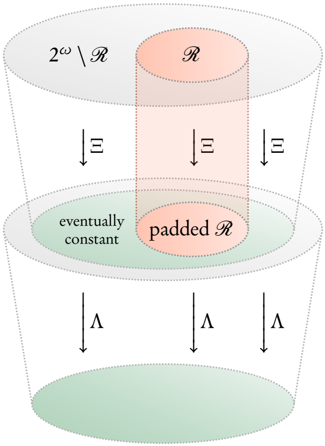
2. For inputs in $\Xi(\mathscr{R})$,
   - φ is run with a “good” oracle, and computes a fast growing function;

A non-trivial diminutive measure



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2. For inputs in $\Xi(\mathscr{R})$,
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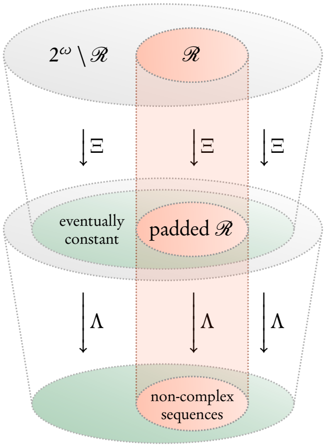
A non-trivial diminutive measure



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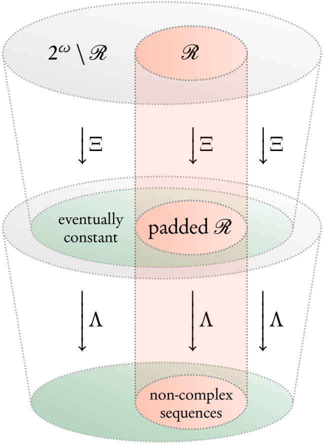
   - While it searches in this way for terminating computations, it keeps outputting blocks of identical bits.
2. For inputs in $\Xi(\mathscr{R})$,
   - φ is run with a "good" oracle, and computes a fast growing function;
   - while waiting for φ to converge the bit blocks will become very long;
   - one can show that this implies that the output is not complex.

A non-trivial diminutive measure



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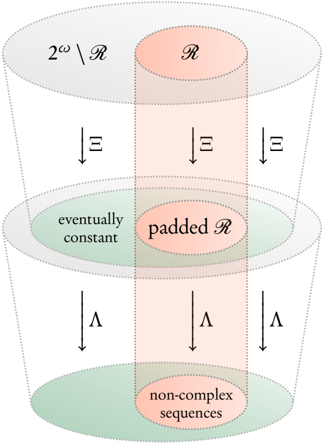
A non-trivial diminutive measure



1. If $X \in \Xi(2^\omega \setminus \mathscr{R})$, by construction, X is eventually constant.
2. Then Λ will find only finitely many oracle bits, and output the same bit forever.

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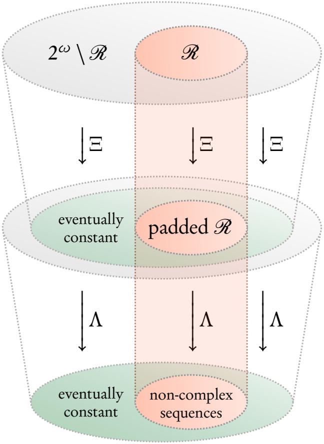

1. If $X \in \Xi(2^\omega \setminus \mathscr{R})$, by construction, X is eventually constant.
2. Then Λ will find only finitely many oracle bits, and output the same bit forever.
3. The same can be forced for $X \notin \Xi(2^\omega)$.

(But this is of no relevance here.)

A non-trivial diminutive measure

1. Now let μ be the measure induced by $\Lambda \circ \Xi$, that is,

$$\mu(\mathscr{Y}) = \lambda\{Z \colon \Lambda \circ \Xi(Z) \in \mathscr{Y}\}$$

   for all $\mathscr{Y} \subseteq 2^\omega$.
2. By the previous arguments, no $X \in \mathrm{MLR}_\mu$ is complex.
3. Then the Proposition implies that μ is diminutive.
4. But every sequence in $\Lambda \circ \Xi(\mathscr{R})$ computes a fast-growing function, so is not computable, so is not an atom.
5. Then since $\mu(\Lambda \circ \Xi(\mathscr{R})) = \lambda(\mathscr{R}) > 0$, we have that $\mu(\mathrm{Atoms}_\mu) < 1$, thus μ is not trivial. $\square$

A known result as an easy corollary

1. **Corollary (Kautz).** There is a computable, non-trivial measure μ such that no Δ^0_2, non-computable $X \in \mathrm{MLR}_\mu$ exists.
2. **Proof.**
   - Non-computable randoms for μ are images of $\mathrm{MLR}^{\emptyset'}$ sequences under $\Lambda \circ \Xi$. Then they are $\mathrm{MLR}^{\emptyset'}$ with respect to μ.
   - Any Δ^0_2 is trivially covered by a μ-Martin-Löf test relative to $\emptyset'$.
   - So no non-computable random for μ can be Δ^0_2. $\square$
3. This new proof is priority-free!

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Thank you for your attention.

APAL 4/2017, pp. 860–886