Randomness for semi-measures

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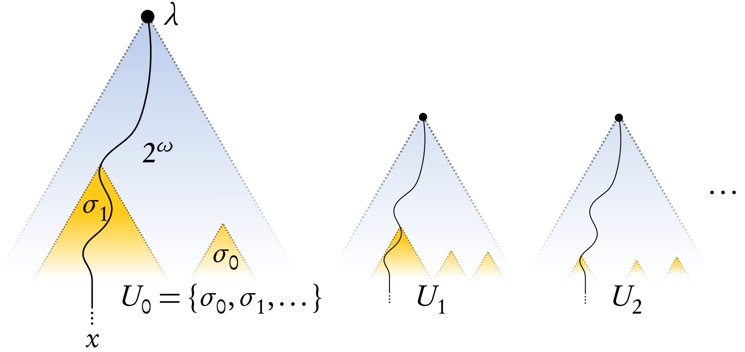
These slides *(now!)* at http://db.tt/61VYqen0

Motivation

1. Algorithmic randomness has been intensively studied for computable and non-computable measures.
2. Algorithmic randomness is closely related to computability theory; most of the work is on the interaction between both fields.
3. For a computability theoretic reason that we will discuss, there is a class of objects similar to measures that is relevant for algorithmic randomness, namely left-c.e. semi-measures.
4. We will try to understand randomness w.r.t. to these objects.

A little history

Martin-Löf randomness



1. **Martin-Löf randomness.** Real is not random if in the intersection of a sequence of uniformly Σ^0_1 classes, whose measure tends to 0 at a guaranteed minimum speed.
2. Classically, the Lebesgue measure is used here.

Computable probability measures on 2^∞

1. **Definition.** A probability measure μ on 2^∞ is *computable* if $\sigma \mapsto \mu([\![\sigma]\!])$ is computable as a real-valued function.
2. Formally, if there is a computable $\hat{\mu} : 2^{<\infty} \times \mathbb{N} \to \mathbb{Q}$ such that

$$|\mu([\![\sigma]\!]) - \hat{\mu}(\sigma, i)| \leq 2^{-i}$$

for every $\sigma \in 2^{<\infty}$ and $i \in \mathbb{N}$.

Martin-Löf randomness for computable measures

1. Let μ be a computable measure.
2. **Definition.** A *μ-Martin-Löf test* is a sequence $(\mathscr{U}_n)_n$ of uniformly Σ^0_1 classes such that for all n, $\mu(\mathscr{U}_n) \leq 2^{-n}$.
3. **Definition.** $X \in 2^\infty$ is called *μ-Martin-Löf random* if for any μ-ML-test $(\mathscr{U}_n)_n$ we have $X \notin \bigcap_n (\mathscr{U}_n)$.

Reminder: Turing functionals

1. **Intuition.** A *Turing functional* effectively converts one infinite binary sequence into another.
2. **Definition.** A *Turing functional* $\Phi : 2^\infty \to 2^\infty$ is a (partial) function for which there exists a Turing machine M such that

$$\sigma, \sigma' \in \mathrm{dom}(M) \ \wedge\ \sigma \sqsubseteq \sigma' \implies M(\sigma) \sqsubseteq M(\sigma')$$

   For A where $M(A \restriction n)$ halts for all n, let $\Phi(A) = \lim_{n\to\infty} M(A \restriction n)$. Otherwise $\Phi(A)$ is undefined.
3. **Definition.** Φ is *almost total* if $\lambda(\mathrm{dom}(\Phi)) = 1$.

Induced measures

1. Let Φ be an almost total Turing functional.
2. **Definition.** The *measure induced by* Φ is

$$\lambda_\Phi(\sigma) = \lambda(\Phi^{-1}(\sigma)) = \lambda\{X \mid \sigma \sqsubset \Phi^X\}.$$

3. **Careful!** If Φ is *not* almost total, this need not be a measure.
4. **Proposition.** Every computable probability measure is induced by an almost total Turing functional.
5. **Theorem.** Φ almost total and $X \in \mathsf{MLR}$ implies $\Phi(X) \in \mathsf{MLR}_{\lambda_\Phi}$.

Randomness for non-computable measures

1. Reimann/Slaman studied random for non-computable measures.
2. There are two different ways of using the non-computability.
3. Of course we always evaluate the measure condition w.r.t. the non-computable measure.
4. But we have a choice of whether the procedure enumerating the test has access to the non-computable measure or not.
5. In the first case, we need to represent the measure somehow as an element of 2^∞, so that the procedure can access it as oracle.
6. This representation will not be unique.
   (as representations of real-valued functions typically are)
7. We will usually be interested in representations as easy as possible w.r.t. Turing reducibility.

Randomness for non-computable measures

1. Let μ be non-computable, and R_μ be a representation of μ.
   - An R_μ*-Martin-Löf test* is a sequence $(\mathscr{U}_i)_{i\in\omega}$ of uniformly $\Sigma^0_1(R_\mu)$ classes with $\mu(\mathscr{U}_i) \leq 2^{-i}$ for all i.
   - X is μ*-Martin-Löf random*, denoted $X \in \mathsf{MLR}_\mu$, if there *exists* some R_μ for μ such that X passes all R_μ-ML-tests.
2. **Intuition.** μ is so "weak" that it can be represented in ways that are computationally too weak to derandomize X.

Blind randomness

1. Some measures are complex enough that *all* of their representations have significant derandomization power.
2. This interferes with randomness.
3. To deal with this, consider *blind* randomness, first studied by Kjos-Hanssen.
   - A *blind μ-Martin-Löf test* is a sequence $(\mathcal{U}_i)_{i\in\omega}$ of uniformly Σ^0_1 classes with $\mu(\mathcal{U}_i) \leq 2^{-i}$ for all i.
   - X is *blind μ-Martin-Löf random*, denoted $X \in \mathsf{bMLR}_\mu$, if X passes *every* blind μ-ML-test.

Left-c.e. semimeasures

Left-c.e. semimeasures

1. A semi-measure is not guaranteed to be additive, but only to be "superadditive".
2. That is, we only have $\rho(\sigma) \geq \rho(\sigma 0) + \rho(\sigma 1)$.
   (We also allow $\rho(\emptyset) \leq 1$.)
3. ρ is called left-c.e. if we can uniformly in the input σ approximate $\rho(\sigma)$ from below.

Induced semi-measures

1. We can again look at induced measures, with the same definition:

$$\lambda_\Phi(\sigma) = \lambda(\Phi^{-1}(\sigma)) = \lambda\{X \mid \sigma \sqsubset \Phi^X\}.$$

2. This time we don't require almost totality; measure loss corresponds to paths where the functional is not defined.
3. **Proposition (Levin/Zvonkin).** Every left-c.e. semi-measure is induced by a Turing functional.
4. So left-c.e. semi-measures directly correspond to Turing functionals, and are therefore natural objects to consider.
5. There is a universal left-c.e. semimeasure, denoted by M.

Randomness for semi-measures: the straight-forward way

1. Naïve definition: Plug in semi-measure instead of measure.
2. This notion behaves strangely.
3. **Proposition (BHPS).** There is a left-c.e. semi-measure ρ such that for any sequence $(\mathcal{U}_i)_{i\in\omega}$ of uniform Σ^0_1 classes we have that

$$(\forall i\colon \rho(\mathcal{U}_i) \leq 2^{-i}) \implies \bigcap_{i\in\mathbb{N}} \mathcal{U}_i = \emptyset.$$

4. In other words, all valid tests are empty.
5. There are no non-randoms.

The less straight-forward approach

1. In 2012 at Dagstuhl, Shen asked:
   - If Φ and Ψ are Turing functionals with $\lambda_\Phi = \lambda_\Psi$, does it follow that $\Phi(\mathsf{MLR}) = \Psi(\mathsf{MLR})$?
2. If yes, we could define:
   - Y is random w.r.t. a semi-measure ρ if for every Φ with $\rho = \lambda_\Phi$ there is some $X \in \mathsf{MLR}$ such that $\Phi(X) = Y$.
3. But it is relatively easy to construct a counterexample.

What we aim for

Some (debatable) desiderata

1. **Coherence:** If X is random with respect to μ *as measure*, we also want X to be random with respect to μ seen *as a semi-measure*.
2. **Randomness preservation:** If $X \in \mathsf{MLR}$ and Φ is a Turing functional, then $\Phi(X)$ is random with respect to λ_Φ.
3. **No randomness from nothing:** If Y is random with respect to the semi-measure λ_Φ for some Turing functional Φ, then there is some $X \in \mathsf{MLR}$ such that $\Phi(X) = Y$.
4. **Computable Sequence Condition:** If X is computable and random with respect to a semi-measure ρ, then $\inf_n \rho(X \upharpoonright n) > 0$.
   - **Motivation.** Reimann and Slaman show that a sequence X is random w.r.t. some measure μ and not an atom of μ iff X is not computable.

Making a measure out of a semi-measure

Two approaches

1. Another idea is to apply randomness definitions for measures to semi-measures.
2. For this we must change the semi-measure into a measure.
3. Intuitively, a semi-measure loses measure along the way down a path. Two opposing approaches to resolve this:
   - Either *increase* the measures of the two children of a node in such way that their sum equals the measure of the parent node,
   - or do the opposite and *decrease* the measure of the parent to the sum of the measures of its two children.
4. Formally, these two approaches are
   - Solomonoff normalization, and
   - the "bar approach" by V'yugin.

Solomonoff normalization

1. The *Solomonoff normalization* of ρ is defined[1] via

$$\widetilde{\rho}(\varnothing) = 1, \quad \widetilde{\rho}(\sigma) = \rho(\sigma) \prod_{i=0}^{|\sigma|-1} \frac{\rho(\sigma \restriction i)}{\rho(\sigma \restriction i^\frown 0) + \rho(\sigma \restriction i^\frown 1)}.$$

2. **Intuition.** Maintain the measure *ratio* between the two children, but scale them up to exhaust the measure made available by the parent.

[1] Unclear how to define the normalisation if a positive measure node is followed by two measure 0 children. But this problem does not occur with the example on the next slide, and that shows that this is not a promising approach anyway.

Solomonoff normalization

1. Unfortunately, this notion behaves quite strangely from a computability point of view.
2. For example we can construct two left-c.e. semi-measures ρ_1 and ρ_2 such that
   - they are "essentially the same" in the sense that they have the same values and every loss of measures of ρ_1 also appears in ρ_2, except sometimes one (!) step further down along a path;
   - but (every representation of) $\widetilde{\rho}_2$ codes the Halting problem, while (some representation of) $\widetilde{\rho}_1$ is computable.

Cutting back a semi-measure

1. So let's try the opposite approach.
2. V'yugin defined $\overline{\rho}(\sigma) := \inf_n \sum_{\tau \succeq \sigma \ \& \ |\tau| = n} \rho(\tau)$.
3. This is the largest measure such that $\overline{\rho} \leq \rho$.
4. For Φ inducing ρ we have $\overline{\rho}(\sigma) = \lambda(\{X : \Phi(X)\downarrow \ \& \ \Phi^X \succ \sigma\})$.
5. Can we use $\overline{\rho}$ to define randomness for ρ?

$\overline{\rho}$ can be complicated

1. **Theorem (BHPS).** The following are equivalent for $\alpha \in (0,1)$.
   - α is $\emptyset'$-right c.e..
   - $\alpha = \limsup_n q_n$ for a computable sequence of rationals $(q_n)_{n\in\omega}$.
   - $\alpha = \inf r_n$ where $(r_n)_{n\in\omega}$ is a sequence of uniformly left-c.e. reals.
   - There is a semi-measure ρ such that $\overline{\rho} = \alpha \cdot \lambda$.
2. In other words, we can make a left-c.e. semi-measure ρ such that (every representation of) $\overline{\rho}$ codes $\emptyset''$.
3. **Proposition (BHPS).** There is a positive $\emptyset'$-computable measure μ with a low representation such that $\mu \neq \alpha \cdot \overline{\rho}$ for every left-c.e. real α and every left-c.e. semi-measure ρ.
4. **Question.** Can we achieve computably dominated?

Blind bar randomness

1. The derandomization power of $\emptyset''$ interferes with randomness.
2. So if we want to define randomness using the bar approach, we should look at the blind version, denoted by $\mathsf{bMLR}_{\overline{\rho}}$.
3. **Proposition (BHPS).** There is a semi-measure ρ such that
   - $\rho = \lambda_\Phi$ for some Turing functional Φ;
   - $\mathrm{dom}(\Phi) \cap \mathsf{MLR} \neq \emptyset$; and
   - $\mathsf{bMLR}_{\overline{\rho}} = \emptyset$.
4. In other words, we have no randomness preservation.

W2R and semi-measures

Weak 2-randomness w.r.t. a semimeasure

1. **Definition.** For a left-c.e. semi-measure ρ, a *generalized ρ-test* is a sequence $(\mathscr{U}_i)_{i\in\omega}$ of uniformly Σ^0_1 classes with $\lim_{i\to\infty} \rho(\mathscr{U}_i) = 0$.
2. So this is the naïve notion of weak 2-randomness w.r.t. a semi-measure.
3. But it behaves well:
   - **Theorem (BHPS).** X passes every generalized ρ-test iff $X \in \mathsf{bW2R}_{\overline{\rho}}$.
4. And we have preservation of randomness!
   - **Theorem (BHPS).** If $X \in \mathsf{W2R} \cap \mathrm{dom}(\Phi)$, then $\Phi(X) \in \mathsf{bW2R}_{\overline{\rho}}$.
5. The Computable Sequence Condition also holds.
6. "No randomness from nothing" holds for truth-table functionals, but is open in general.

Conclusion

Intimidating diagram

$$
\begin{array}{ccccccc}
\mathsf{2MLR}_{\rho} & \subsetneq & \mathsf{W2R}_{\rho} & \subsetneq & \mathsf{AR}_{\rho} & \subsetneq & \mathsf{MLR}_{\rho} \\
\rotatebox{-90}{$\subseteq$} & & \parallel & & \rotatebox{-90}{$\nsubseteq$} & & \rotatebox{-90}{$\nsubseteq$} \\
\mathsf{b2MLR}_{\overline{\rho}} & \subsetneq & \mathsf{bW2R}_{\overline{\rho}} & \subsetneq & \mathsf{bAR}_{\overline{\rho}} & \subseteq & \mathsf{bMLR}_{\overline{\rho}} \\
\rotatebox{-90}{$\nsubseteq$} & & \rotatebox{-90}{$\nsubseteq$} & & \rotatebox{-90}{$\nsubseteq$} & & \rotatebox{-90}{$\nsubseteq$} \\
\mathsf{2MLR}_{\overline{\rho}} & \subsetneq & \mathsf{W2R}_{\overline{\rho}} & \subsetneq & \mathsf{AR}_{\overline{\rho}} & = & \mathsf{MLR}_{\overline{\rho}}
\end{array}
$$

Open questions

1. **Question.** If Φ and Ψ are Turing functionals such that $\lambda_\Phi(\sigma) = \lambda_\Psi(\sigma)$ for every $\sigma \in 2^{<\infty}$, does it follow that $\Phi(\mathsf{W2R}) = \Psi(\mathsf{W2R})$?
   - We know this is wrong for MLR, but holds for 2-random.
   - It also holds for W2R for truth-table functionals.
2. **Question.** If ρ is a left-c.e. semi-measure, does $\overline{\rho}$ have a least Turing degree representation?
3. **Question.** Does $\overline{M}$ have a least Turing degree representation?
4. **Question (Hoyrup).** Do we get a reasonable notion of randomness for a semi-measure by defining a sequence X to be random with respect to ρ if

$$X \in \bigcup_{\{\Phi:\rho=\lambda_\Phi\}} \Phi(\mathsf{MLR})?$$

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Thanks for your attention.