Randomness in the Weihrauch degrees

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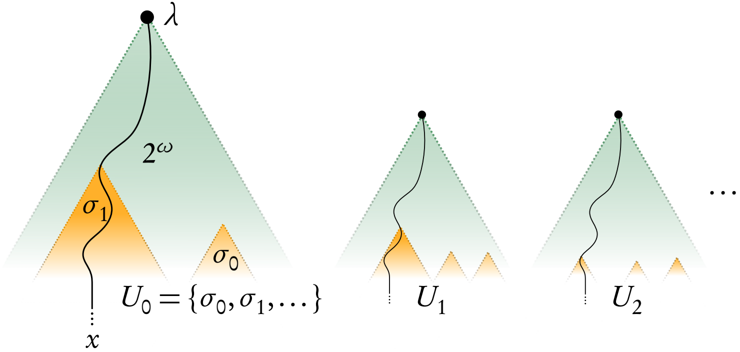
Joint work with Paul Shafer

These slides *(now!)* at https://db.tt/bECt6WnC

Paradigm

“What can be computed with access to randomness?”

Formal definition of randomness



1. **Martin-Löf randomness.** Real is not random if in the intersection of an effective sequence of c.e. classes, whose measure tends to 0 at a guaranteed minimum speed.

A first answer: The Kučera-Gács-Theorem

1. **Theorem.** For *every* sequence A there is a Martin-Löf random sequence X with $A \leq_{\mathrm{T}} X$.
2. **But this is not satisfactory.**
   - We are not really using the randomness of X.
   - If we did, then every other, “equally random” Y would work.
   - But in fact only a measure 0 set works.
   - In the proof X is “cherry-picked”.
3. **What we really want.** We want to know what we can compute given access to *any* sufficiently random sequence.

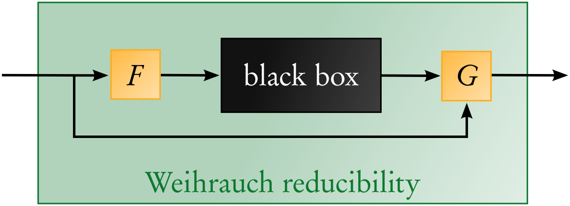
A second answer: Sacks' theorem

1. **Theorem.** If A is computable from every X in a set of positive measure, then A is computable.
2. **This is unsatisfactory as well.**
    - We do not want to compute a single sequence A.
    - We want to know what *mathematical tasks* can be done with randomness.
    - For a given task there may be many allowed solutions; this breaks the majority vote mechanic in the proof of Sacks' theorem.

Therefore: Weihrauch degrees

1. **General idea.** We look at mathematical tasks where a problem is given to a black box, and the black box has to give back a solution to the problem.
2. **Example.** Given a bounded sequence of rationals, the black box has to produce an accumulation point.
3. **Reducibility.** Assume we have a black box solving a certain problem $\mathscr{A}$. Can we use it to solve another problem $\mathscr{B}$?
4. **Approach.**
   - Code an instance B of problem $\mathscr{B}$ into a valid instance A of problem $\mathscr{A}$.
   - Run the black box for $\mathscr{A}$ on A.
   - Convert the solution for A back into a solution for B.

Formally: Weihrauch degrees



1. F and G are both Turing functionals.
2. **Weak reducibility.** The decoding procedure G has access to the original input.
3. That is, if B is the black box, the computed function is

$$x \longmapsto G(\langle B(F(x)), x\rangle).$$

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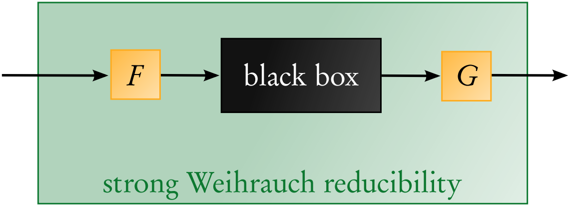

1. F and G are both Turing functionals.
2. **Strong reducibility.** The decoding procedure G *does not* have access to the original input.
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$$x \mapsto G(B(F(x))).$$

The Weihrauch degree of MLR

1. The function MLR in the Weihrauch degrees is defined as

$$\begin{aligned} \mathrm{MLR}\colon \quad 2^\omega &\rightrightarrows \mathit{MLR} \\ A &\longmapsto \mathit{MLR}^A. \end{aligned}$$

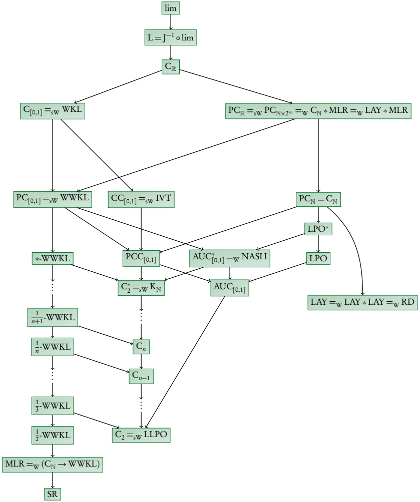
   (non-italic = the principle MLR; *italic* = set of MLR sequences)

2. If the input is computable this is a black box giving us an MLR object.
3. Using the reducibility defined before, we can now ask what can be reduced to this “principle”.

The Weihrauch zoo

1. The reducibilities produce a degree structure of mathematical tasks of different levels of difficulty.

The Weihrauch zoo



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$\mathrm{L} = \mathrm{J}^{-1} \circ \lim$

$\mathrm{C}_{\mathbb{R}}$

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$\mathrm{PC}_{\mathbb{R}} =_{\mathrm{sW}} \mathrm{PC}_{\mathbb{N}\times 2^\omega} =_{\mathrm{W}} \mathrm{C}_{\mathbb{N}} * \mathrm{MLR} =_{\mathrm{W}} \mathrm{LAY} * \mathrm{MLR}$

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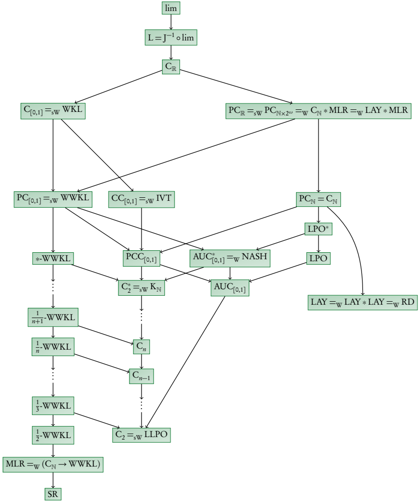

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SR

1. The reducibilities produce a degree structure of mathematical tasks of different levels of difficulty.
2. Many tasks are equivalent to *choice principles*.

Choice principles

1. A choice principle is a black box solving a task as follows:
    - A set is described by giving more and more negative information, encoded into an infinite sequence in, say, $\mathbb{N}^{\mathbb{N}}$.
    - That is, step by step we obtain more and more information which elements are *not* in the set.
    - The task is to give an element that will never be removed.
2. **Example.** Choice on the natural numbers $\mathrm{C}_{\mathbb{N}}$.

Choice principles

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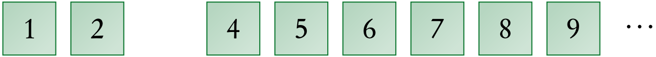
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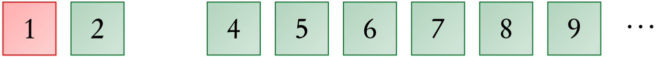

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3. Similarly, we can define $C_{[0,1]}$, $C_{2^{\mathbb{N}}}$, $C_{\mathbb{R}}$, $C_{\mathbb{N}^{\mathbb{N}}}$,…

Positive choice

1. We will want to look at probabilistic computation, which can be modeled by a special kind of choice principle.
2. We will call a choice principle a *positive choice principle* if for every valid instance of the problem there exists positive measure many correct solutions.
3. Of course, the measure will depend on the space on which we need to make a choice.
4. We will use the most canonical measures for every space, but other reasonable choices produce equivalent results.
5. We will look at $\mathrm{PC}_{[0,1]}$, $\mathrm{PC}_{2^{\mathbb{N}}}$, $\mathrm{PC}_{\mathbb{R}}$, $\mathrm{PC}_{\mathbb{N}^{\mathbb{N}}}, \ldots$

More choice

1. $\mathrm{AUC}_{[0,1]}$ is *all or unique choice* on $[0,1]$, that is, either every point in the unit interval is an acceptable answer; or only a single real.
2. Clearly, here the challenge is only to know whether *anything* is ever removed or not.
3. For any principle $\mathscr{P}$ in the Weihrauch degrees, define

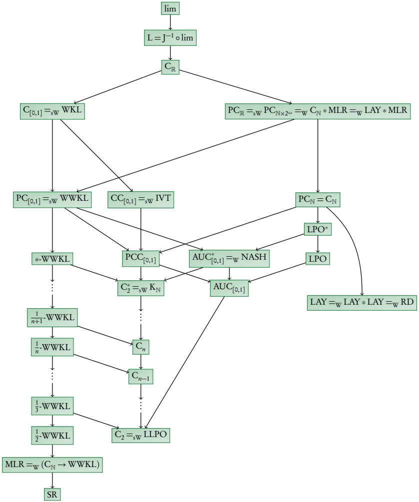
   $$\mathscr{P}^* := \bigsqcup_{i \in \mathbb{N}} \mathscr{P}^i,$$

   that is, $\mathscr{P}^*$ solves *any finite product* of instances for $\mathscr{P}$.
4. Pauly showed that $\mathrm{AUC}^*_{[0,1]}$ is equivalent to computing Nash equilibria.

WWKL as randomized computation

1. *Weak weak König's Lemma* (WWKL) is the principle that, given a closed set $P \subseteq 2^\omega$ of positive measure, returns a path of P.
2. Reduction to WWKL can be seen as a type of randomized computation, namely computation by a Las Vegas machine.
3. A Las Vegas machine is a machine succeeding with positive probability and such that it will detect if one of the randomized computations fails.

Two types of randomized computation



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SR

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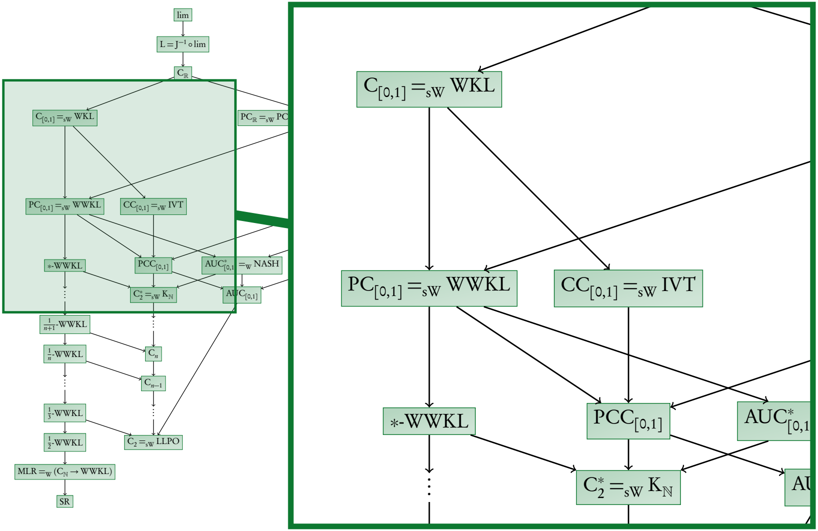

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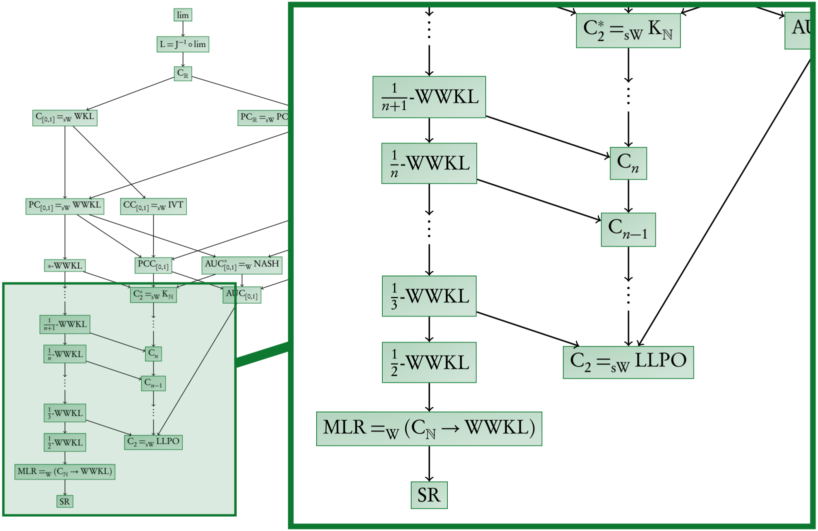


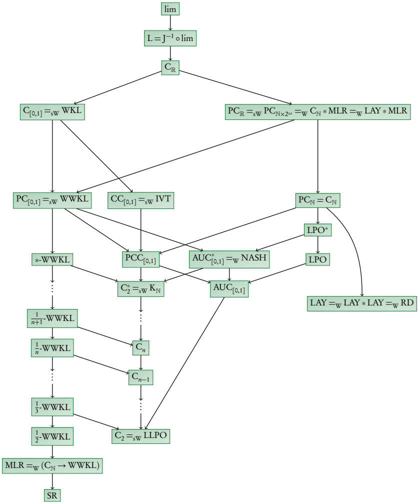
A contrast with reverse maths

1. **Therefore:** In the Weihrauch degrees, Las Vegas machines can do much more than MLR oracles.
    - For example, they can compute Nash equilibria while MLR cannot.
2. In reverse maths, the principle WWKL also exists, and has been intensively studied.
3. Ditto for the principle MLR, that states that for every existing set, there also exists a set MLR relative to it.
4. In this setting it turns out that MLR equals WWKL (shown essentially by Kučera).

Origin of the different behavior

1. So in the fine-grained Weihrauch degrees we can see a difference between two models of randomized computation.
2. This difference is lost in the degree structure of reverse maths.
3. The difference between both settings is that in reverse maths...
    - ... we can use a principle arbitrarily often in a proof, and
    - ... we can use non-uniform arguments.

Layerwise computability (Hoyrup & Rojas) in the lattice



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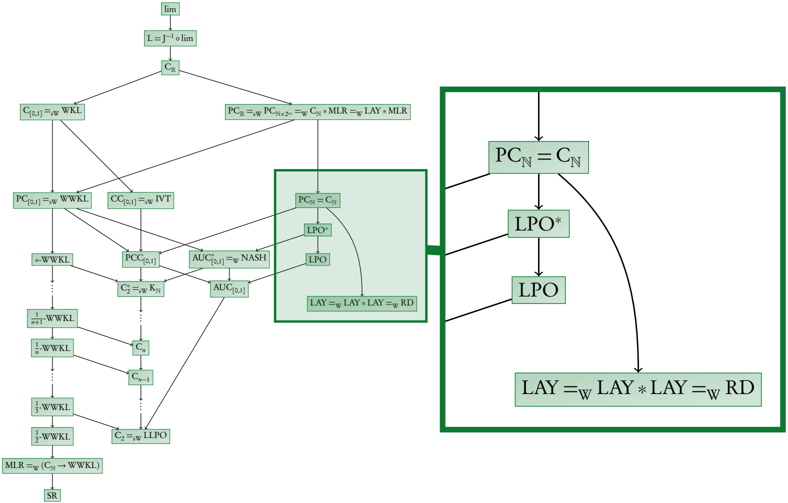

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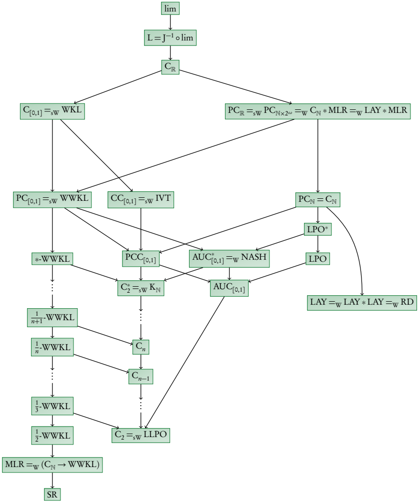
Other principles interacting with randomness



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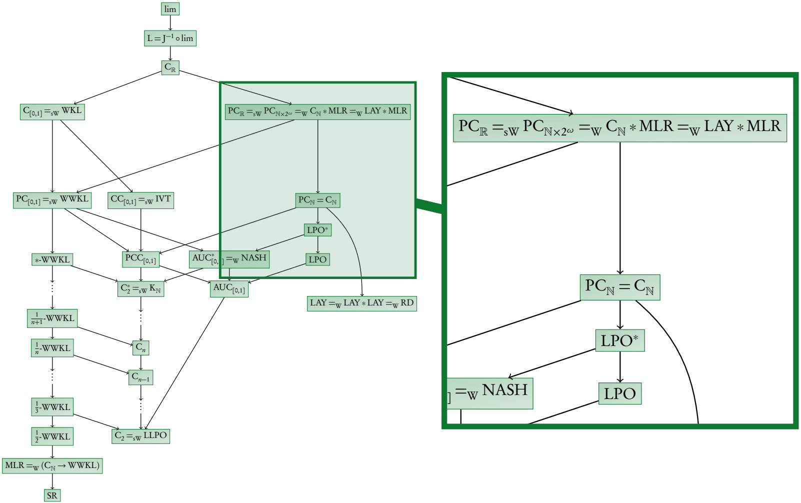
$C_2 =_{sW} \mathrm{LLPO}$

$\frac{1}{2}$-WWKL

$\mathrm{MLR} =_W (C_{\mathbb{N}} \to \mathrm{WWKL})$

SR

Other principles interacting with randomness



lim

$L = J^{-1} \circ \lim$

$C_{\mathbb{R}}$

$C_{[0,1]} =_{sW} WKL$

$PC_{\mathbb{R}} =_{sW} PC_{\mathbb{N}\times 2^{\omega}} =_{W} C_{\mathbb{N}} * MLR =_{W} LAY * MLR$

$PC_{[0,1]} =_{sW} WWKL$

$CC_{[0,1]} =_{sW} IVT$

$PC_{\mathbb{N}} = C_{\mathbb{N}}$

LPO^*

$*$-WWKL

$PCC_{[0,1]}$

$AUC^*_{[0,1]} =_{W} NASH$

LPO

$C_2^* =_{sW} K_{\mathbb{N}}$

$AUC_{[0,1]}$

$LAY =_{W} LAY * LAY =_{W} RD$

$\frac{1}{n+1}$-WWKL

C_n

$\frac{1}{n}$-WWKL

C_{n-1}

$\frac{1}{3}$-WWKL

$C_2 =_{sW} LLPO$

$\frac{1}{2}$-WWKL

$MLR =_{W} (C_{\mathbb{N}} \to WWKL)$

SR

$PC_{\mathbb{R}} =_{sW} PC_{\mathbb{N}\times 2^{\omega}} =_{W} C_{\mathbb{N}} * MLR =_{W} LAY * MLR$

$PC_{\mathbb{N}} = C_{\mathbb{N}}$

LPO^*

$] =_{W} NASH$

LPO

Some work in progress

1. 2-randomness: MLR $*_s$ lim
   - Not yet clear how much structure of the lattice is preserved when applying the jump.
2. The Vitali Covering Lemma
   - Brown, Giusto, Simpson: In reverse maths equivalent to WWKL.
   - In the context of Weihrauch degrees we get a much richer picture, depending on the formalization.
   - One version is computable; another is equivalent to WWKL; yet another strictly stronger than WWKL.
   - We do not know the precise location in the lattice of that last version yet.
3. Other notions, theorems, etc. from algorithmic randomness in this framework.

Thank you for your attention.