Randomness versus superspeedability

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Based on joint work with Philip Janicki, Wolfgang Merkle, and Frank Stephan

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Motivation

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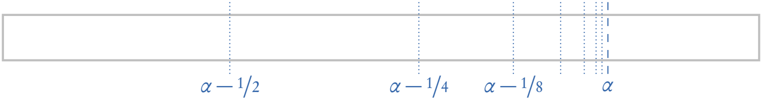

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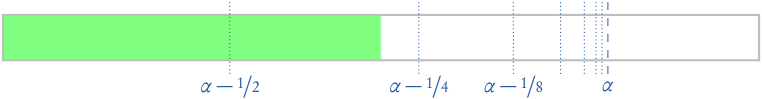

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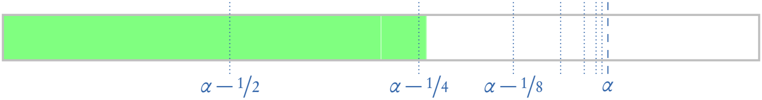

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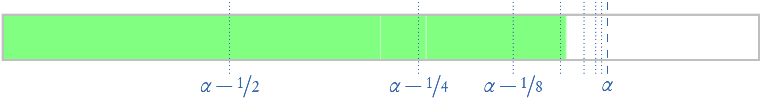

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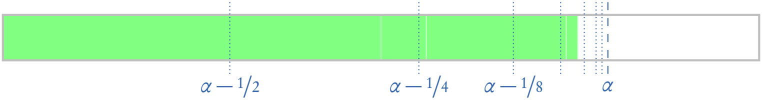

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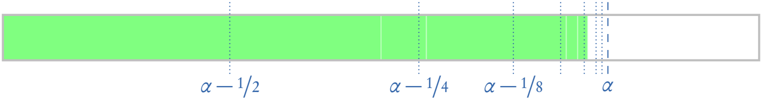

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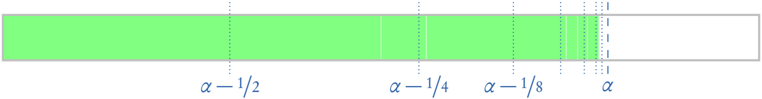

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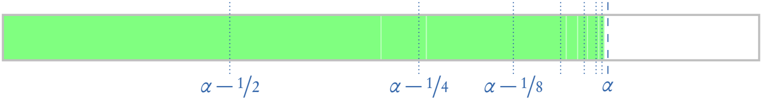

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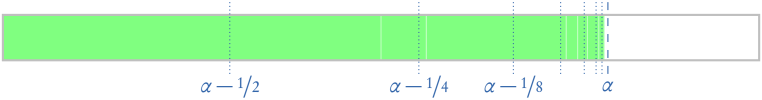

4. So far, so trivial.

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2. We are interested in numbers having "benign" approximations
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5. Finally, we inquire into the relationship with randomness.

2 Background

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5. Weaker algorithmic "pattern exploitation mechanisms" define weaker randomness notions, such as *Schnorr randomness*.

Speedability

Definition (Merkle & Titov). α is *speedable* if there is

- a $\rho \in (0, 1)$ and
- a computable left-approximation $(a_n)_n$ of α

such that there are infinitely many $n \in \mathbb{N}$ with

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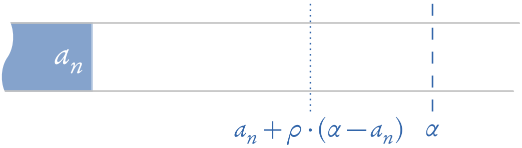

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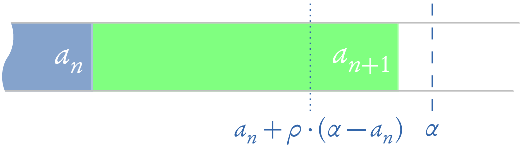

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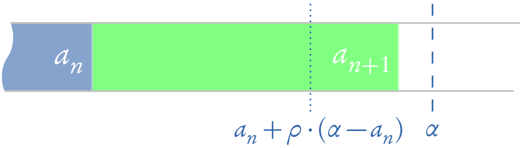

2 **Theorem (Merkle & Titov).** Any $\rho \in (0,1)$ works equally.

(But you need to nonuniformly replace the approximation by another one.)

Speedability & randomness

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2. **Question (Merkle & Titov).**
   - Does the inverse hold?
   - **That is:** Among the left-computables, are the randoms characterized by their non-speedability?

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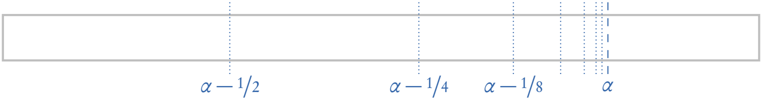

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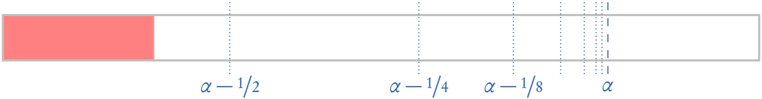

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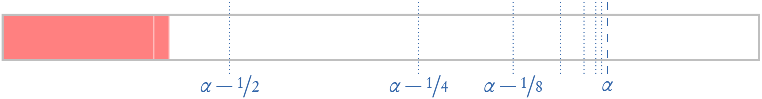

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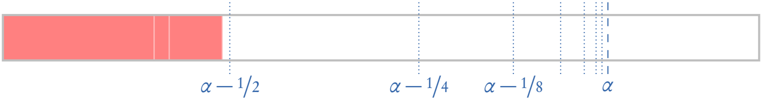

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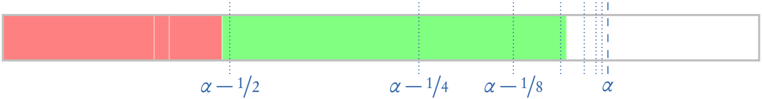

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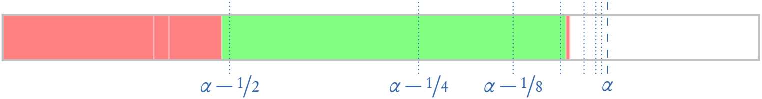

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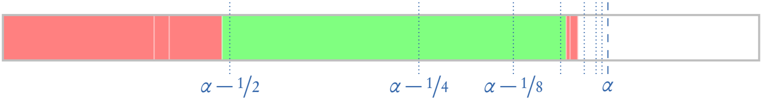

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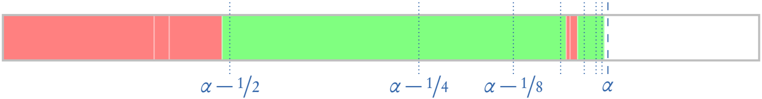

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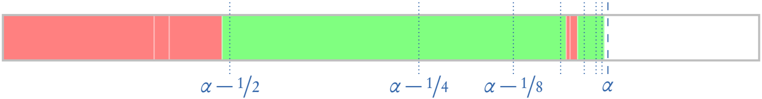

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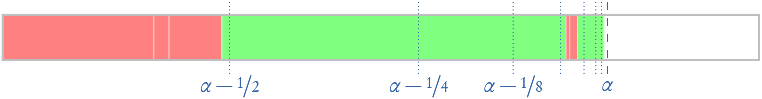

3. **Intuition.**
   - As in general left-computability, the approximation can "dawdle" arbitrarily, *but* infinitely often it must "catch up" to how fast computable numbers can be approximated.
   - Obviously, (in general) we do not know when these good moments occur; in case we do, α is again computable.

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2. Thus, with Peter Hertling, we studied many of their properties.
3. **Relevant for this talk in particular:**
   - **Theorem (Hertling, Hölzl & Janicki).**
     The regainingly approximable numbers lie properly between the computable and the left-computable ones.

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3. **Answer.** Almost, but not quite. Even a small jump could be the one finally catching up, if a large jump was made previously.
4. But still, catching up requires making some big jump *somewhere*, and we can prove the following statement as a consequence.
5. **Proposition.** Every regainingly approximable α is speedable.

The converse is not true

1. **Proposition (Merkle & Titov).** Every left-computable α that is the binary expansion of a c.e. set is speedable.
2. **Theorem (Hertling, Hölzl & Janicki).** Not all such α are regainingly approximable.

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2. Thus, making a number non-regainingly approximable looks easier than making it non-speedable.
3. But of course, due to the last slide, this is not good enough to negatively answer the open question yet.

So what?

1. The notion of regaining approximability requires something to have happened at some specific time.
2. Thus, making a number non-regainingly approximable looks easier than making it non-speedable.
3. But of course, due to the last slide, this is not good enough to negatively answer the open question yet.

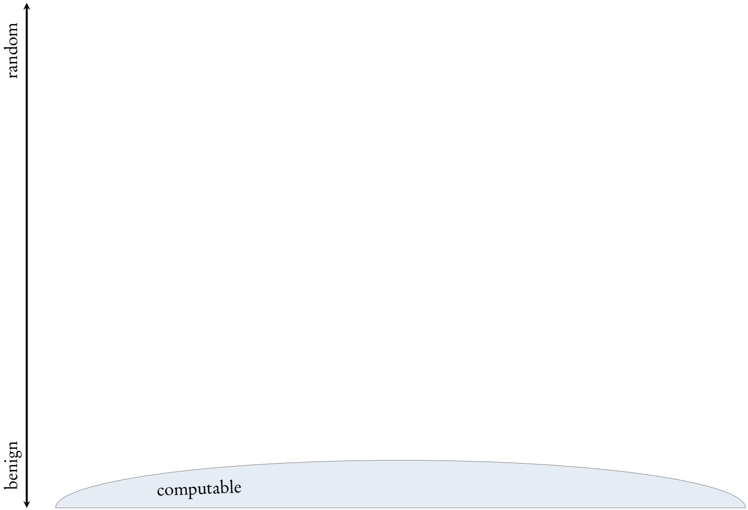
4. However, a very complex argument involving all above notions of benign approximation (as well as one not mentioned here) shows:
   - **Theorem (Hölzl & Janicki).** The answer to the above question is **no**; randoms are not the only left-computable non-speedables.

3 New contributions

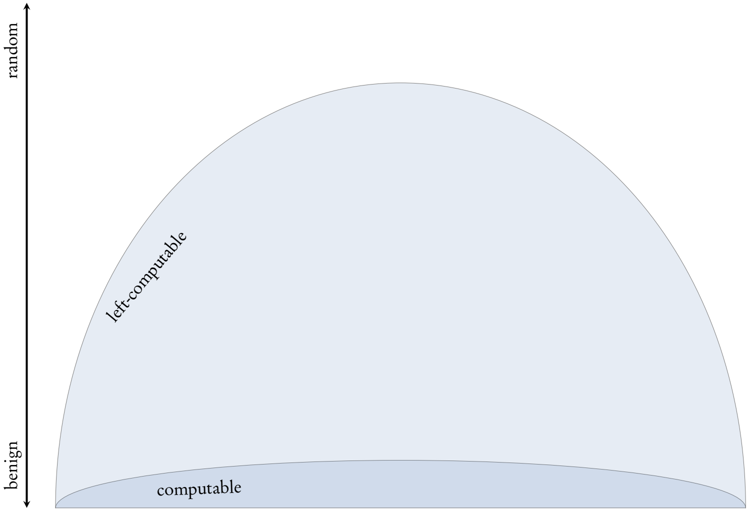
Benignness versus randomness

Benignness versus randomness

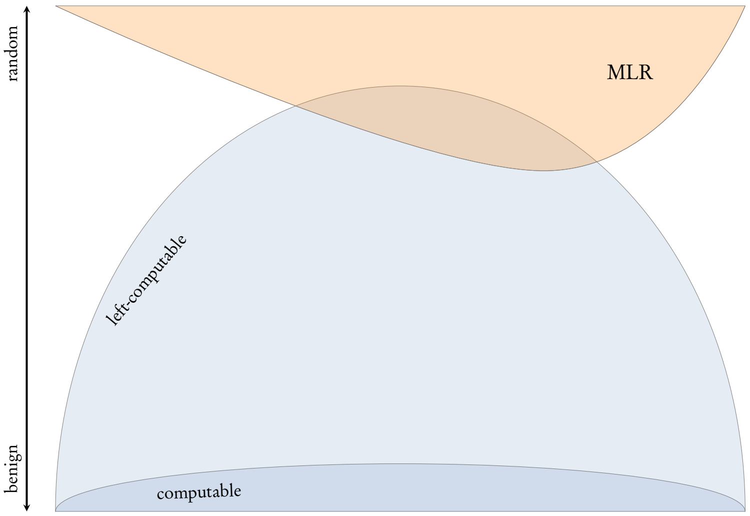
Benignness versus randomness



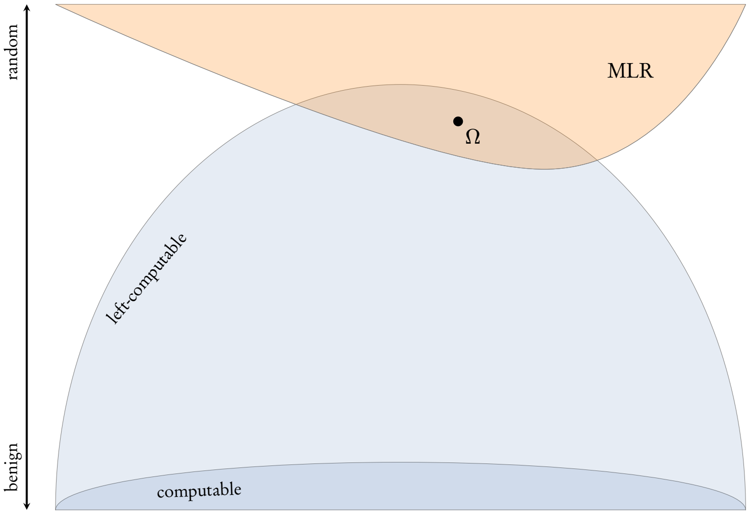
Benignness versus randomness



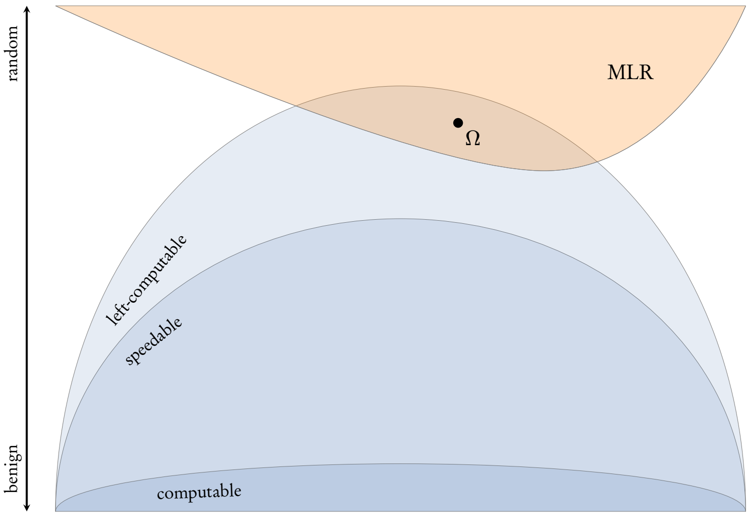
Benignness versus randomness



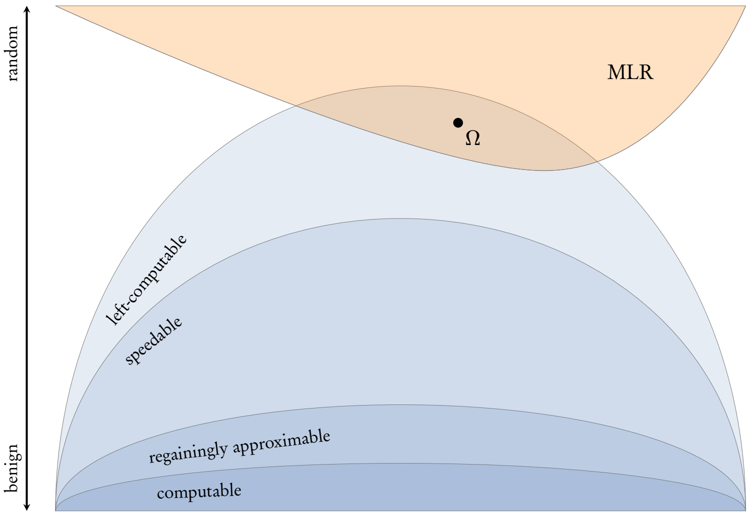
Benignness versus randomness



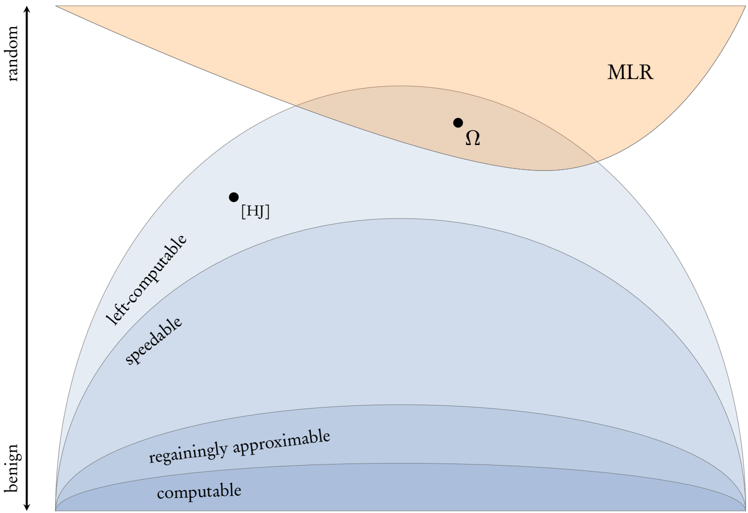
Benignness versus randomness



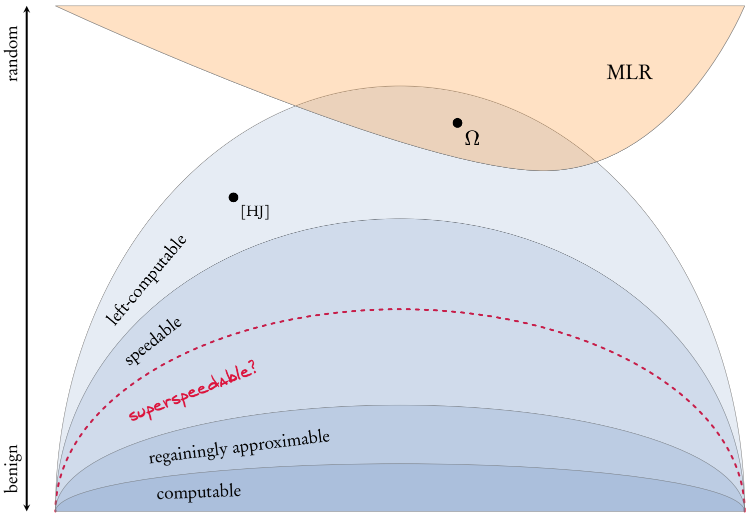
Benignness versus randomness



Benignness versus randomness



Benignness versus randomness



1 **Question (Barmpalias).** Do all speedables have a *single* approximation whose ρ goes to 1? Or is that a smaller set?

Superspeedability?

1. **Definition.** We call α *superspeedable* if there is a computable left-approximation $(a_n)_n$ of α such that
$$\limsup_{n\to\infty} \frac{a_{n+1} - a_n}{\alpha - a_n} = 1.$$
2. **Question (Barmpalias).** Is speedable = superspeedable?

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2. **Question (Barmpalias).** Is speedable = superspeedable?

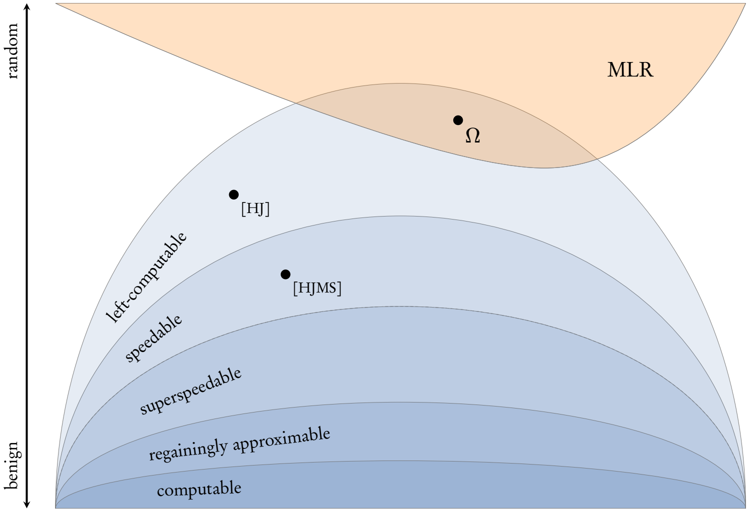
3. **Theorem (Titov?).** For left-computable numbers,

   not immune implies speedable.

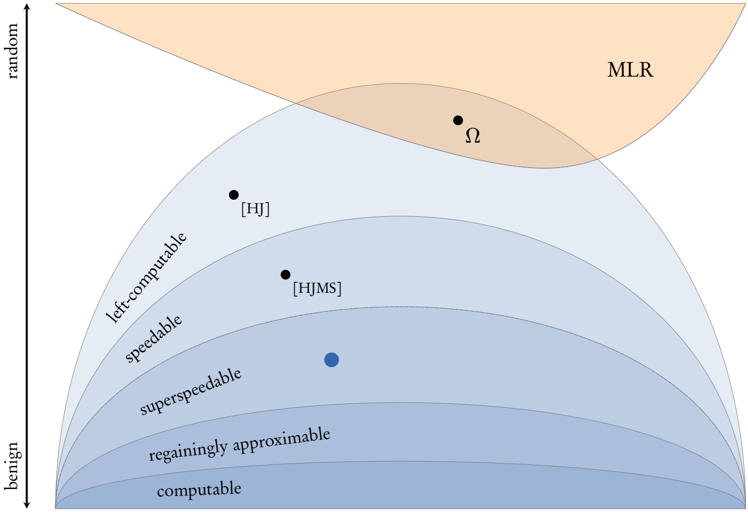
4. **Theorem.** There exists a left-computable number that is

   not immune and not superspeedable.

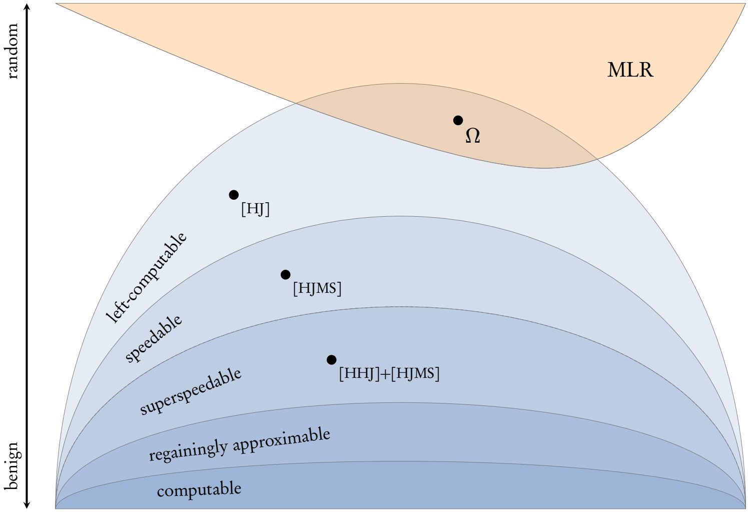
Benignness versus randomness



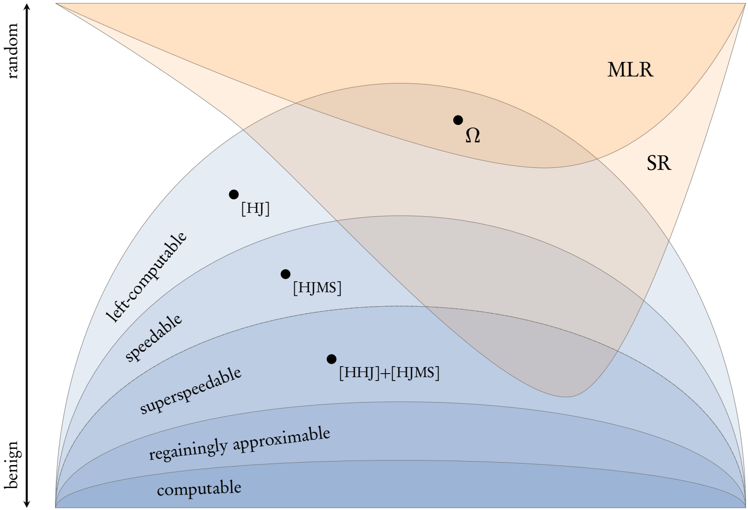
Benignness versus randomness



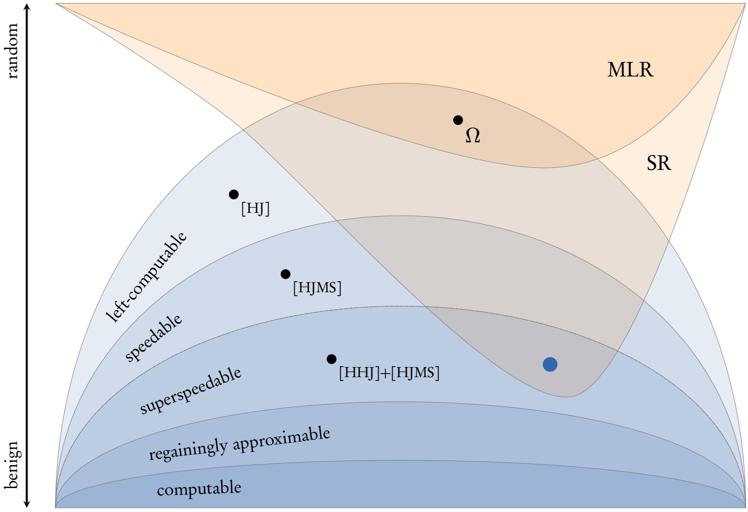
Benignness versus randomness



Benignness versus randomness



Benignness versus randomness



A superspeedable Schnorr random

1 **Theorem.** There is a superspeedable Schnorr random.

A superspeedable Schnorr random

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2. **Theorem (Franklin, Stephan).** The following are equivalent.
   - A is not Schnorr random;
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$$\exists^\infty n \, (d(A \upharpoonright f(n)) \geqslant n).$$

(That is, there is a computable lower bound on the winning speed of d.)

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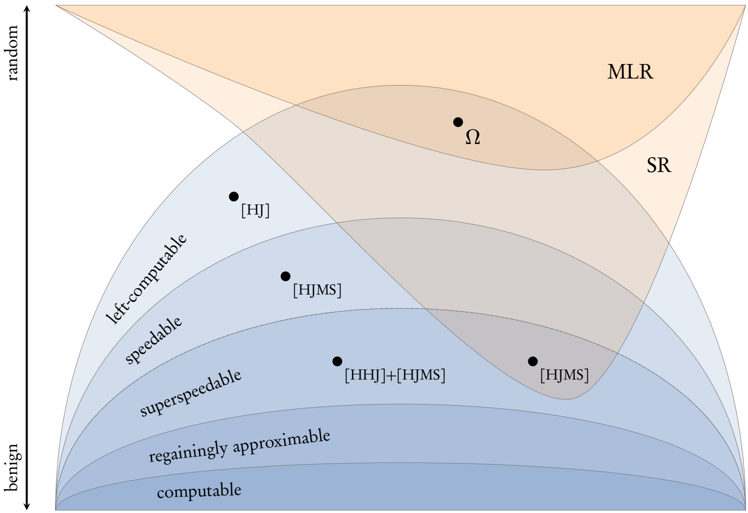
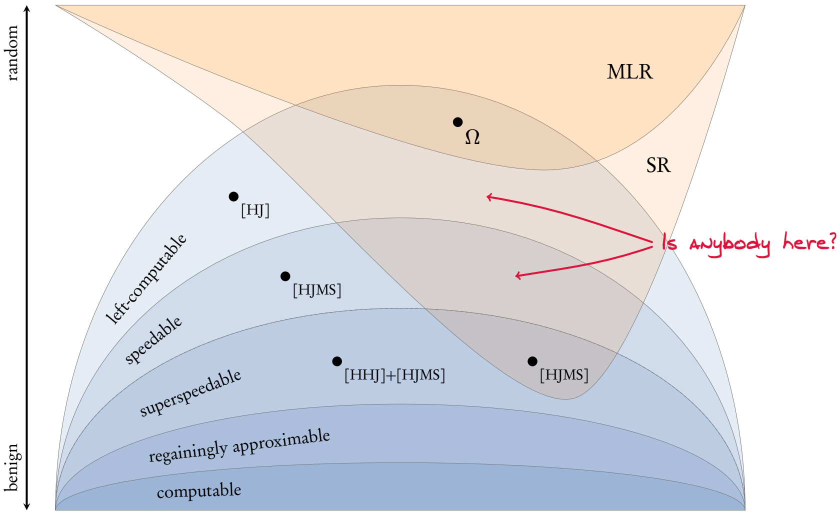
3. Our contruction works by constructing a superspeedable A that gives the possible d's so few chances at winning any money, that no f as above can be computable.

Summary


random
benign
MLR
SR
Ω
[HJ]
left-computable
[HJMS]
speedable
superspeedable
[HHJ]+[HJMS]
[HJMS]
regainingly approximable
computable

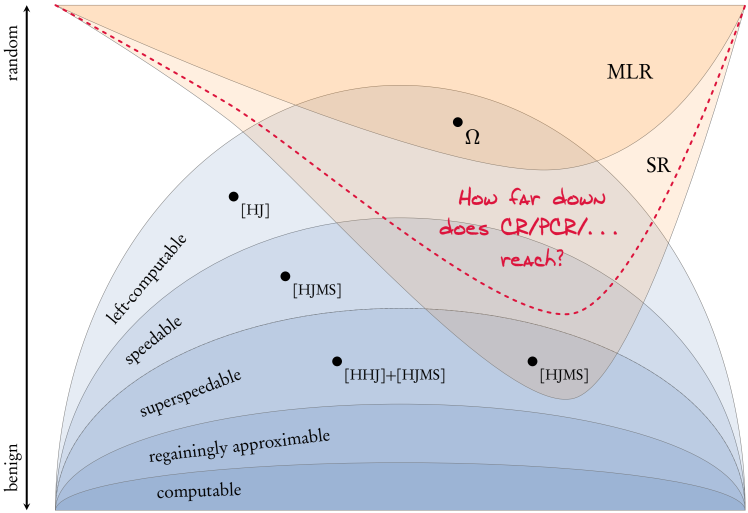
Selected open questions



1 **Open question.** Do numbers in the marked fields exist?

(It would be strange if not, but we have not constructed any so far.)

Selected open questions



2 **Open question.** How benignly approximable can computable randoms, partial computable randoms, weak s-randoms,… be?

What AI makes of "Bratislava," "speed," and "randomness."


Thank you for your attention!
10.4230/LIPIcs.MFCS.2024.62
What AI makes of "Bratislava," "speed," and "randomness."