

Rank and randomness

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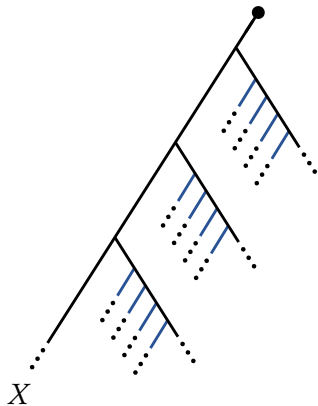
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Motivation

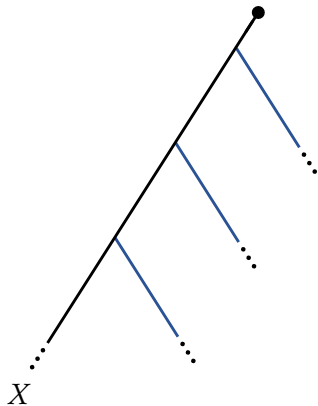
“The Cantor-Bendixon rank $\text{rk}_{\mathcal{P}}(X)$ of a sequence X inside a Π_1^0 class $\mathcal{P} \subseteq 2^\omega$ measures how tightly integrated X is into \mathcal{P} .”

- 1 Formally:** Iterate the operation of removing all isolated sequences from \mathcal{P} . Then (if it exists) $\text{rk}_{\mathcal{P}}(X)$ is the number of the last iteration after which X is still present.
- 2** A result of Kreisel implies that always $\text{rk}_{\mathcal{P}}(X) < \omega_1^{\text{CK}}$.

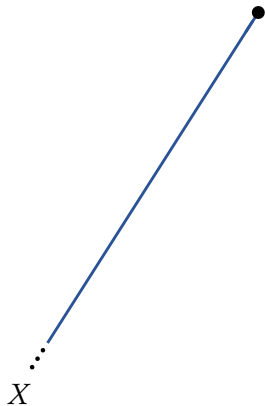
Examples



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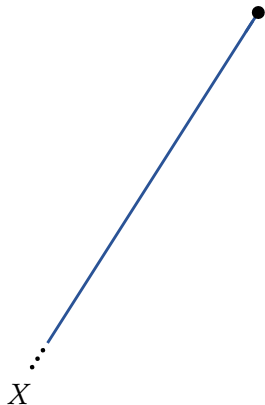


Examples



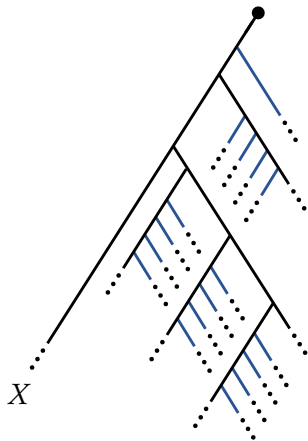
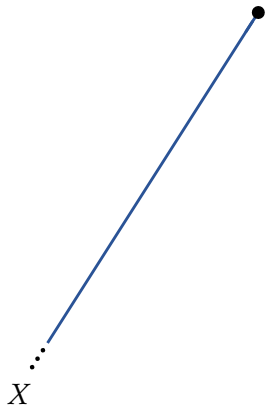
X is now isolated

Examples



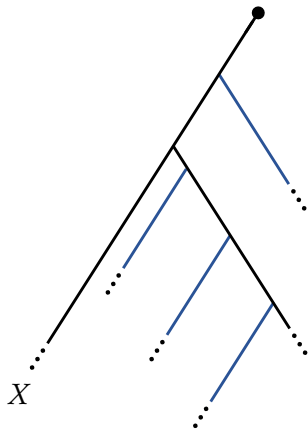
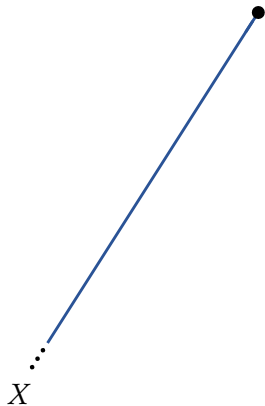
X is now isolated
 $\Rightarrow \text{rk}_{\mathcal{P}}(X) = 2.$

Examples



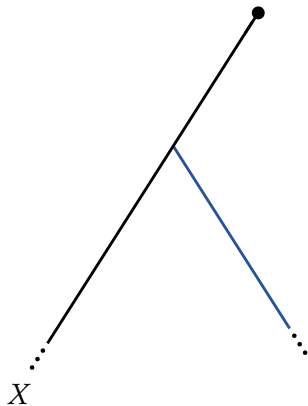
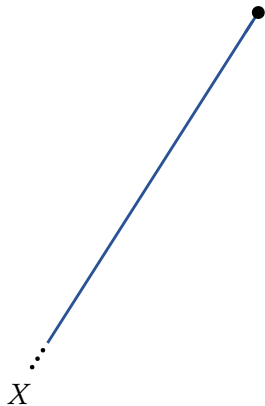
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Examples



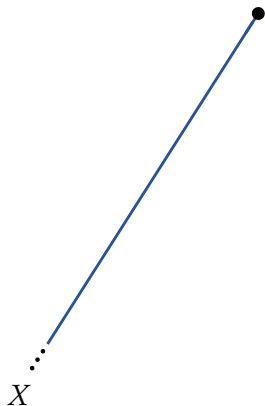
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Examples

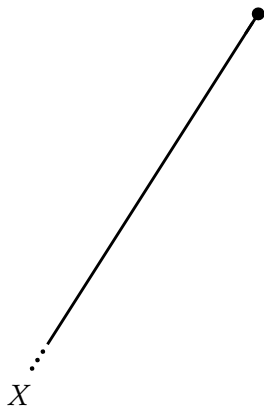


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Examples

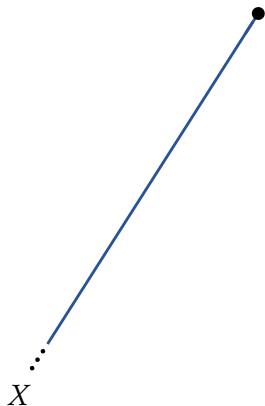


X is now isolated
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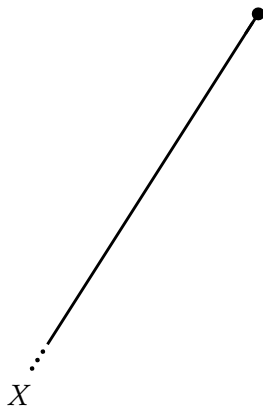


X is still not isolated

Examples



X is now isolated
 $\Rightarrow \text{rk}_{\mathcal{P}}(X) = 2.$



X is still not isolated
 $\Rightarrow \text{rk}_{\mathcal{P}}(X) = \omega.$

Rank and computability

1 Definition. If a Π_1^0 class \mathcal{P} exists where $\text{rk}_{\mathcal{P}}(X)$ is defined, then

$$\text{rk}(X) = \min\{\alpha : \mathcal{Q} \text{ is a } \Pi_1^0 \text{ class with } \text{rk}_{\mathcal{Q}}(X) = \alpha\}.$$

Otherwise $\text{rk}(X)$ is undefined.

- 2 Theorem (Cenzer, Smith).** For all computable ordinals $\alpha > 0$ there is some Δ_2^0 degree \mathbf{a} such that some $X \in \mathbf{a}$ has $\text{rk}(X) = \alpha$.
- 3 Theorem (Cenzer, Smith).** Every Δ_2^0 degree has a rank 1 point.
- 4 Theorem (Downey, Cholak).** For every c.e. degree \mathbf{c} and every computable ordinal $\alpha > 0$, there is a c.e. $C \in \mathbf{c}$ with $\text{rk}(C) = \alpha$.
- 5 Theorem (Downey, Wu, Yang).** For every Δ_2^0 degree \mathbf{a} and every computable ordinal $\alpha > 0$, exists an $X \in \mathbf{a}$ with $\text{rk}(X) = \alpha$.
- 6 Question.** Is there a relation between rank and *randomness*?

- 1 Proposition (Porter, Bienvenu).** There is an X with $\text{rk}(X) = 1$ such that X is random with respect to a computable measure.
- 2 Proposition (Porter).** In every Δ_2^0 random degree exists an X random with respect to a computable measure with $\text{rk}(X) = 2$.
- 3 Our goal:** A stronger and more general result, analogous to that of Downey/Wu/Yang, but for randomness.

Our main result

1 **Definition.** The *support* of a measure μ is the set

$$\{X \in 2^\omega : \forall n \mu(X \upharpoonright n) > 0\}.$$

2 **Definition.** A Π_1^0 class \mathcal{P} is *rank-faithful* if $\text{rk}(X) = \text{rk}_{\mathcal{P}}(X)$ for all $X \in \mathcal{P}$ such that $\text{rk}(X)$ exists.

3 **Theorem.** For each Δ_2^0 random degree \mathbf{r} and each computable ordinal $\alpha > 0$, there is a countably supported computable measure μ and a sequence $R \in \mathbf{r}$ such that

- R is random with respect to μ ,
- R has Cantor-Bendixson rank α , and
- the support of μ is rank-faithful.

4 Can be seen as a “randomized version” of Downey/Wu/Yang.

5 Additionally, the last point has a strong implication...

The relevance of rank-faithfulness

- 1 Let μ be a computable measure with rank-faithful support \mathcal{P} .
- 2 Let $(\mathcal{U}_i)_{i \in \omega}$ be a universal μ -test and $\mathcal{K}_i = 2^\omega \setminus \mathcal{U}_i$ for all $i \in \omega$.
- 3 Clearly $\mathcal{K}_i \subseteq \mathcal{P}$ for every $i \in \omega$.
- 4 Then $\text{rk}_{\mathcal{K}_i}(R) \leq \text{rk}_{\mathcal{P}}(R)$ for every $i \in \omega$ and $R \in \mathcal{K}_i$.
- 5 Since $\text{rk}(R) = \text{rk}_{\mathcal{P}}(R)$ by the rank-faithfulness of \mathcal{P} , we have $\text{rk}_{\mathcal{K}_i}(R) = \text{rk}(R)$ for every $i \in \omega$ such that $R \in \mathcal{K}_i$.
- 6 This implies for every $i \in \omega$ with $R \in \mathcal{K}_i$, that R 's rank $\text{rk}(R) = \text{rk}_{\mathcal{K}_i}(R)$ can already be observed inside \mathcal{K}_i .
- 7 Since \mathcal{K}_i contains only μ -random sequences, the entire structure of sequences that give R its (minimal) rank consists of μ -randoms.
- 8 So not only have we built a μ -random R of desired rank, but the rank of R can be witnessed entirely by sequences that are μ -random themselves (and which inductively have the same property).

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Proving the finite case

The base case $\alpha = 1$

- 1 Let $A \in \mathbf{r}$ be random and $(A_s)_{s \in \omega}$ a Δ_2^0 -approximation of A .
- 2 Without loss of generality let $A_s \neq A_{s+1}$ for every $s \in \omega$.
- 3 For $X \in 2^\omega$, define an X -computable $(s_n^X)_{n \geq 1}$ via
 - s_1^X is the least s such that $X \upharpoonright 1 = A_s \upharpoonright 1$, and
 - for $n \geq 1$, by s_{n+1}^X is the least $s > s_n^X$ with $X \upharpoonright (n+1) = A_s \upharpoonright (n+1)$, if these values exist.
- 4 Next define $\sigma_1^X, \sigma_2^X, \dots$ via

$$\sigma_n^X = \begin{cases} 1^{s_n^X} \frown 0 & \text{if } s_n^X \downarrow, \\ \uparrow & \text{otherwise.} \end{cases}$$

The base case $\alpha = 1$

- 1 Finally define the Turing functional $\Phi_A(X)$ via

$$\Phi_A(X) = \begin{cases} \sigma_1^X \sigma_2^X \dots & \text{if } \forall i \geq 0 : \sigma_i^X \downarrow, \\ \sigma_1^X \sigma_2^X \dots \sigma_{i-1}^X 1^\omega & \text{if } i \text{ is least such that } \sigma_i^X \uparrow. \end{cases}$$

- 2 The measure induced by Φ_A is clearly countably supported.
- 3 It's easy to see that $\Phi_A(A) \equiv_T A$, so $\Phi_A(A) \in \mathbf{r}$.
- 4 It's also immediate that for $X \neq A$, $\Phi_A(X)$ is computable.
- 5 All $\Phi_A(X)$ with $X \neq A$ are isolated, so $\text{rk}_{\Phi_A(2^\omega)}(\Phi_A(X)) = 0$.
- 6 There are infinitely many $\Phi_A(X)$ branching off $\Phi_A(A)$, so $\text{rk}_{\Phi_A(2^\omega)}(\Phi_A(A)) = 1$.
- 7 It remains to show that $\Phi_A(2^\omega)$ is rank-faithful.

From $\alpha = 1$ to $\alpha < \omega$

1 Claim. $\text{rk} = \text{rk}_{\Phi_A(2^\omega)}$.

Proof. $\Phi_A(A)$ is non-computable, so there is no Π_1^0 class \mathcal{Q} in which $\Phi_A(A)$ is isolated, so $\text{rk}(\Phi_A(A)) = 1 = \text{rk}_{\Phi_A(2^\omega)}(\Phi_A(A))$.
For $\Phi_A(X)$ with $X \neq A$, trivially $\text{rk}(X) = \text{rk}_{\Phi_A(2^\omega)}(X) = 0$.

2 Note how this argument only works for $\alpha = 1$!

3 This completes the proof for $\alpha = 1$. □

4 To obtain the case $\alpha < \omega$, interleave finitely many functionals:

$$\Psi_A(X) = \bigoplus_{i=0}^{n-1} \Phi_{A_i}(X_i) \text{ where } A = \bigoplus_{i=0}^{n-1} A_i \text{ and } X = \bigoplus_{i=0}^{n-1} X_i$$

5 Then we (essentially) need to argue

- why this produces the needed ranks $\text{rk}_{\Psi_A(2^\omega)}$,
- why $\Psi_A(2^\omega)$ is rank-faithful.

The case $\alpha < \omega$

- 1 Note that if some $X_i \neq A_i$, then $\Phi_{A_i}(X_i)$ becomes computable.
- 2 If X disagrees from A in some components, then $\Psi_A(X)$ branches off of $\Psi_A(A)$, with some components becoming computable sequences.
- 3 If another sequence Y
 - initially agrees with X for a long enough time so that $\Psi_A(Y)$, branches off of $\Psi_A(A)$ at the same location as $\Psi_A(X)$
 - but then starts disagreeing from A in 1 *more* component than X , then $\Psi_A(Y)$ has 1 more computable component than $\Psi_A(X)$.
- 4 If we keep repeating this, eventually only computable components are left, resulting in a rank 0 point.
- 5 Infinitely many such rank 0 points branch off of some $\Psi_A(Z)$ (that has 1 non-computable component), giving it rank 1.
- 6 This way, the rank structure is built up to the top $\Psi_A(A)$, e.g. the $\Psi_A(Y)$'s witness the 1 higher rank of $\Psi_A(X)$, and so on.

The case $\alpha < \omega$

- 1 It remains to show the rank-faithfulness.
- 2 We use induction over n , the number of interleaved functionals.
- 3 The components of the input and the output are Turing-equivalent component-wise, that is, $\Phi_{A_i}(A_i) \equiv_T A_i$ for each i .
- 4 **Theorem (Owings).** $\text{rk}(X \oplus Y) = \text{rk}(X)$ implies $Y \leq_T X$.
- 5 But as A was random, all A_i are pairwise relatively random.
- 6 So using induction we obtain that the output rank increases by 1 with the addition of every new component A_i . \square

3

Proving the case $\alpha = \omega$

The challenge

- 1 The naïve approach (of interleaving infinitely many functionals) fails.
- 2 **Reason:** $\text{rk}(\Psi_A(A))$ would not exist, as there would be continuum many ways for a sequence to have the wrong information about at least one subsequence A_i of A .
- 3 **Example.** For each $S \subseteq \omega$, there is some $Y^S = \bigoplus_{i \in \omega} Y_i^S \in 2^\omega$ such that $Y_i^S \neq A_i$ if and only if $i \in S$.
- 4 The support of the induced measure would contain a perfect set.

The idea

- 1 We need a process that leads to infinite interleaving only if the input is A , but which otherwise results in finite interleaving.
- 2 The finite number of interleaved sequences should depend on how many components of the input sequence agree with A .

The solution: the dynamic join functional Ξ_A

1 Partition ω into the sets

- $I_0 = \{n \in \omega : n \equiv 0 \pmod{2}\},$
- $I_1 = \{n \in \omega : n \equiv 1 \pmod{4}\},$
- $I_2 = \{n \in \omega : n \equiv 3 \pmod{8}\},$
- ⋮

and let $J_0 = \omega$ and $J_{n+1} = \overline{I_0 \cup \dots \cup I_n}$ for all $n \in \omega$.

- 2 The dynamic join functional Ξ_A on input X launches $\Phi_{A_0}(X_0)$ and writes its output bits into places in $J_0 = \omega$ of the output.
- 3 Once $\Phi_{A_0}(X_0)$ “has made progress” (= outputs a 0), it is restricted to outputting its bits into the unused places of I_0 .
- 4 Then $\Phi_{A_1}(X_1)$ is launched in parallel, outputting its bits into so far unused locations of J_1 .
- 5 Once both $\Phi_{A_0}(X_0)$ and $\Phi_{A_1}(X_1)$ have made progress, $\Phi_{A_1}(X_1)$ is restricted to I_1 , and $\Phi_{A_2}(X_2)$ is launched writing into J_2 , etc.

The solution: the dynamic join functional

- 1 Note how the construction ensures that all places of the output sequence become defined, independently of whether all parallel processes continue to make progress or some get stuck.
- 2 If $X = A$, then all processes will always make more progress, and all countably many functionals will become launched, and all will contribute to the output sequence.
- 3 If $X \neq A$ then the last process that is launched is determined by what's the first component of X that disagrees with A (and by how soon this disagreement is found).

Properties of the dynamic join

- 1 If during the computation of $\Xi_A(X)$ the n -th process is the last that is started, then $\Xi_A(X)$ is essentially a join of $\Phi_{A_i}(X_i)$, $i \leq n$, (with the bits initially distributed in a somewhat complicated fashion).
- 2 We still have that the input and output of Ξ are Turing-equivalent *individually per component*, as for $\alpha < \omega$. (But *how many* components have been encoded into $\Xi(X)$ depends on X .)
- 3 It's easy to see that Ξ_A is total.

Proof of the case $\alpha = \omega$

- 1 Assume for $X \neq A$, X agrees with A in exactly n components.
- 2 Then $\Xi_A(X)$ agrees in exactly n components with $\Xi_A(A)$.
- 3 In particular, $\Xi_A(X)$ has rank n .
(The argument is essentially the same as for $\alpha < \omega$; the finitely many “out of place” bits at the start of each component have no bearing on the infinitary notion of rank.)
- 4 Since now $\Xi_A(A)$ has sequences branching off it whose ranks are all $< \omega$ but converge to ω , $\Xi_A(A)$ itself must have rank ω .
- 5 Finally, rank-faithfulness is essentially shown as for $\alpha < \omega$.

4

Proving the case $\alpha < \omega_1^{\text{CK}}$

Proof of the general case $\alpha < \omega_1^{\text{CK}}$

- 1 Proof by effective transfinite induction.
- 2 Some overhead is needed for notations for computable ordinals.
- 3 Otherwise the proof is similar:
 - The successor case is similar to the inductive step for $\alpha < \omega$.
 - The limit case is similar to the proof for the case $\alpha = \omega$.

- 1 **Theorem, restated.** For each Δ_2^0 random degree \mathbf{r} and each computable ordinal $\alpha > 0$, there is a countably supported computable measure μ and a sequence $R \in \mathbf{r}$ such that
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Thank you for your attention.

arxiv.org/abs/1707.00378