Rank and randomness

Rupert Hölzl
Bundeswehr University Munich

Joint work with Christopher P. Porter
Drake University, Des Moines

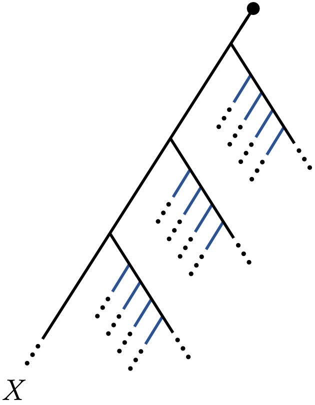
1 Motivation

Cantor-Bedixon rank of a sequence

"The Cantor-Bendixson rank $\mathrm{rk}_{\mathscr{P}}(X)$ *of a sequence* X *inside a* Π^0_1 *class* $\mathscr{P} \subseteq 2^\omega$ *measures how tightly integrated* X *is into* $\mathscr{P}$*."*

1. **Formally:** Iterate the operation of removing all isolated sequences from $\mathscr{P}$. Then (if it exists) $\mathrm{rk}_{\mathscr{P}}(X)$ is the number of the last iteration after which X is still present.

Examples



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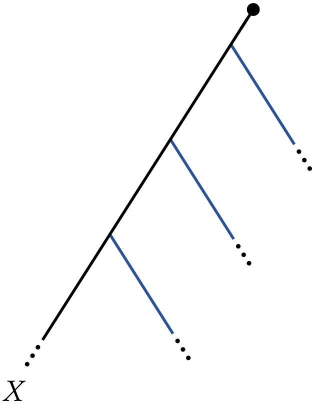

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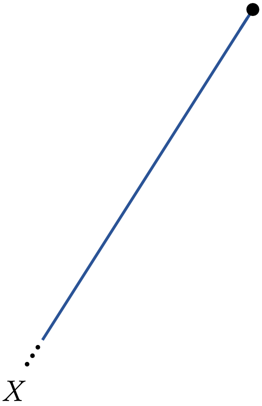

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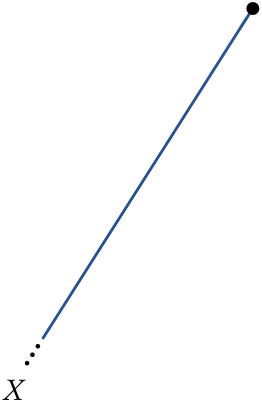

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$\Rightarrow \mathrm{rk}_{\mathscr{P}}(X) = 2.$

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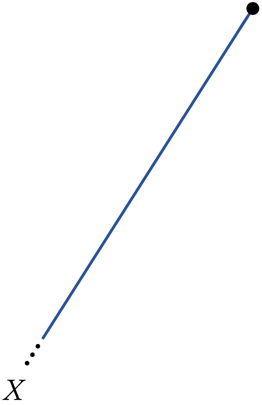

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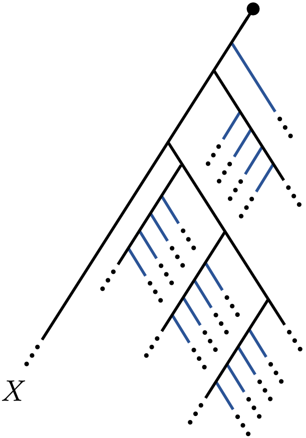

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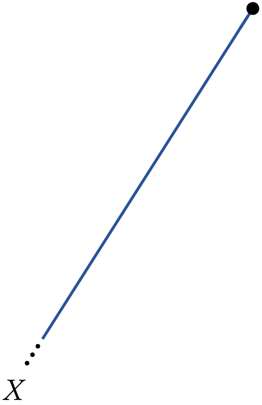

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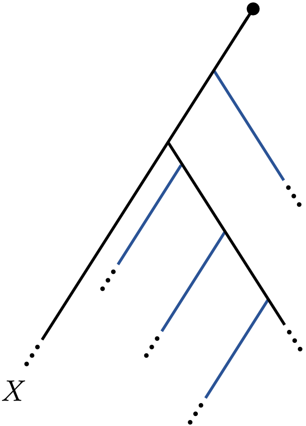

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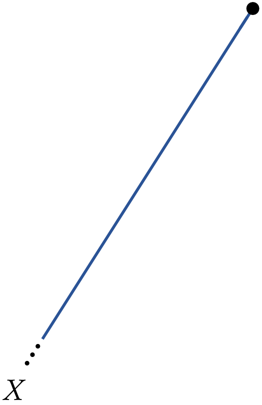

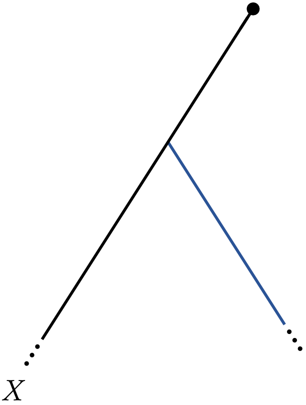

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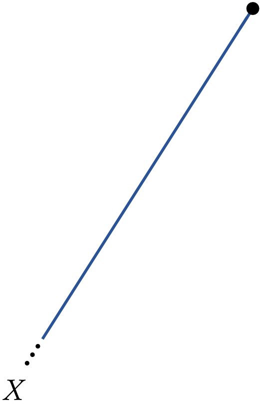

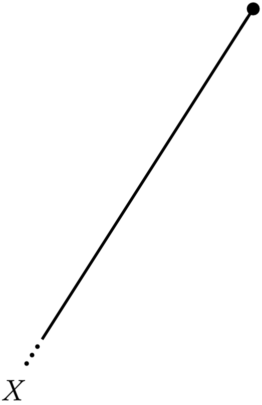

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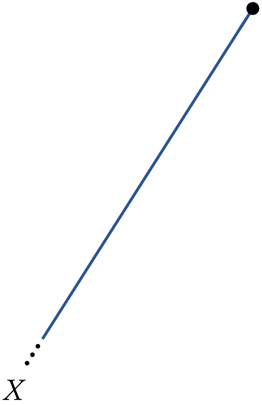

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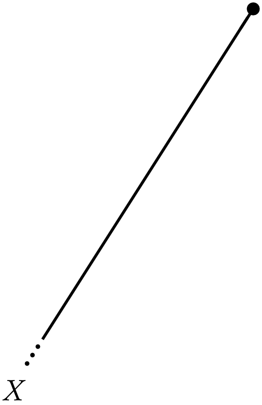

X is still not isolated
$\Rightarrow \mathrm{rk}_{\mathscr{P}}(X) = \omega.$

Rank and computability

1. **Definition.** If a Π^0_1 class $\mathscr{P}$ exists where $\mathrm{rk}_{\mathscr{P}}(X)$ is defined, then

$$\mathrm{rk}(X) = \min\{\alpha : \mathscr{Q} \text{ is a } \Pi^0_1 \text{ class with } \mathrm{rk}_{\mathscr{Q}}(X) = \alpha\}.$$

   Otherwise $\mathrm{rk}(X)$ is undefined.
2. By a result of Kreisel, if it exists, $\mathrm{rk}(X)$ is a computable ordinal.
3. **Theorem (Cenzer, Smith).** For all computable ordinals $\alpha > 0$ there is some Δ^0_2 degree $\mathbf{a}$ such that some $X \in \mathbf{a}$ has $\mathrm{rk}(X) = \alpha$.
4. **Theorem (Cenzer, Smith).** Every Δ^0_2 degree has a rank 1 point.
5. **Theorem (Downey, Wu, Yang).** For every Δ^0_2 degree $\mathbf{a}$ and every computable ordinal $\alpha > 0$, exists an $X \in \mathbf{a}$ with $\mathrm{rk}(X) = \alpha$.

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6. **Question.** Is there a relation between rank and *randomness*?

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3. **Proposition (Porter, Bienvenu).** There is an X with $\mathrm{rk}(X) = 1$ such that X is random with respect to a computable measure.
   (Note that such a measure must have atoms, but that X cannot be one of them.)
4. **Proposition (Porter).** In every Δ^0_2 random degree exists an X random with respect to a computable measure with $\mathrm{rk}(X) = 2$.

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5. **Our goal:** A stronger and more general result, analogous to that of Downey/Wu/Yang, but for randomness.

Our main result

1. **Definition.** The *support* of a measure μ is the set
$$\{X \in 2^\omega : \forall n\ \mu(X \upharpoonright n) > 0\}.$$
2. **Definition.** A Π^0_1 class $\mathscr{P}$ is *rank-faithful* if $\mathrm{rk}(X) = \mathrm{rk}_{\mathscr{P}}(X)$ for all $X \in \mathscr{P}$ such that $\mathrm{rk}(X)$ exists.
3. **Theorem.** For each Δ^0_2 random degree $\mathbf{r}$ and each computable ordinal $\alpha > 0$, there is a countably supported computable measure μ and a sequence $R \in \mathbf{r}$ such that
   - R has Cantor-Bendixson rank α,
   - R is random with respect to μ, and
   - the support of μ is rank-faithful.
4. Can be seen as a "randomized version" of Downey/Wu/Yang.
5. Additionally, the last point has a strong implication...

The relevance of rank-faithfulness

1. Let $\mathscr{P}$ denote the rank-faithful support of μ.
2. Let $(\mathscr{U}_i)_{i\in\omega}$ be a universal μ-test and $\mathscr{K}_i = 2^\omega \setminus \mathscr{U}_i$ for all $i \in \omega$.
3. Clearly $\mathscr{K}_i \subseteq \mathscr{P}$ for every $i \in \omega$.
4. Then $\mathrm{rk}_{\mathscr{K}_i}(R) \leq \mathrm{rk}_{\mathscr{P}}(R)$ for every $i \in \omega$ and $R \in \mathscr{K}_i$.
5. Since $\mathrm{rk}(R) = \mathrm{rk}_{\mathscr{P}}(R)$ by the rank-faithfulness of $\mathscr{P}$, we have $\mathrm{rk}_{\mathscr{K}_i}(R) = \mathrm{rk}(R)$ for every $i \in \omega$ such that $R \in \mathscr{K}_i$.
6. This implies for every $i \in \omega$ with $R \in \mathscr{K}_i$, that R's rank $\mathrm{rk}(R) = \mathrm{rk}_{\mathscr{K}_i}(R)$ can already be observed inside $\mathscr{K}_i$.
7. Since $\mathscr{K}_i$ contains only μ-random sequences, the entire structure of sequences that give R its (minimal) rank consists of μ-randoms.
8. So not only have we built a μ-random R of desired rank, but the rank of R can be witnessed entirely by sequences that are μ-random themselves (and which inductively have the same property).

2 Proving the finite case

The base case $\alpha = 1$

1. Let $A \in \mathbf{r}$ be random and $(A_s)_{s\in\omega}$ a Δ^0_2-approximation of A.
2. Without loss of generality let $A_s \neq A_{s+1}$ for every $s \in \omega$.
3. For $X \in 2^\omega$, define an X-computable $(s_n^X)_{n\geq 1}$ via
   - s_1^X is the least s such that $X\restriction 1 = A_s\restriction 1$, and
   - for $n \geq 1$, by s_{n+1}^X is the least $s > s_n^X$ with $X\restriction(n+1) = A_s\restriction(n+1)$,

   if these values exist.
4. Next define $\sigma_1^X, \sigma_2^X, \ldots$ via

$$\sigma_n^X = \begin{cases} 1^{s_n^X}{}^\frown 0 & \text{if } s_n^X\downarrow, \\ \uparrow & \text{otherwise.} \end{cases}$$

The base case $\alpha = 1$

1. Finally define the Turing functional $\Phi_A(X)$ via

$$\Phi_A(X) = \begin{cases} \sigma_1^X \sigma_2^X \cdots & \text{if } \forall i \geq 0 : \sigma_i^X \downarrow, \\ \sigma_1^X \sigma_2^X \cdots \sigma_{i-1}^X 1^\omega & \text{if } i \text{ is least such that } \sigma_i^X \uparrow. \end{cases}$$

2. The measure μ induced by Φ_A is clearly countably supported.
3. It's easy to see that $\Phi_A(A) \equiv_\mathrm{T} A$, so $\Phi_A(A) \in \mathbf{r}$.
4. It's also immediate that for $X \neq A$, $\Phi_A(X)$ is computable.
5. All $\Phi_A(X)$ with $X \neq A$ are isolated, so $\mathrm{rk}_{\Phi_A(2^\omega)}(\Phi_A(X)) = 0$.
6. There are infinitely many $\Phi_A(X)$ branching off $\Phi_A(A)$, so $\mathrm{rk}_{\Phi_A(2^\omega)}(\Phi_A(A)) = 1$.
7. It remains to show that $\Phi_A(2^\omega)$ is rank-faithful.

From $\alpha = 1$ to $\alpha < \omega$

1. **Claim.** $\mathrm{rk} = \mathrm{rk}_{\Phi_A(2^\omega)}$.

   Proof. $\Phi_A(A)$ is non-computable, so there is no Π^0_1 class $\mathcal{Q}$ in which $\Phi_A(A)$ is isolated, so $\mathrm{rk}(\Phi_A(A)) = 1 = \mathrm{rk}_{\Phi_A(2^\omega)}(\Phi_A(A))$. For $\Phi_A(X)$ with $X \neq A$, trivially $\mathrm{rk}(X) = \mathrm{rk}_{\Phi_A(2^\omega)}(X) = 0$.
2. **Note how this argument only works for $\alpha = 1$!**
3. This completes the proof for $\alpha = 1$. $\square$

From $\alpha = 1$ to $\alpha < \omega$

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4. To obtain the case $\alpha < \omega$, interleave finitely many functionals:

$$\Psi_A(X) = \bigoplus_{i=0}^{n-1} \Phi_{A_i}(X_i) \text{ where } A = \bigoplus_{i=0}^{n-1} A_i \text{ and } X = \bigoplus_{i=0}^{n-1} X_i$$

5. Then we (essentially) need to argue
   - why this produces the needed ranks $\mathrm{rk}_{\Psi_A(2^\omega)}$,
   - why $\Psi_A(2^\omega)$ is rank-faithful.

The case $\alpha < \omega$

1. Note that if some $X_i \neq A_i$, then $\Phi_{A_i}(X_i)$ becomes computable.
2. If X disagrees from A in some components, then $\Psi_A(X)$ branches off of $\Psi_A(A)$, with some components becoming computable sequences.
3. If another sequence Y
   - initially agrees with X for a long enough time so that $\Psi_A(Y)$, branches off of $\Psi_A(A)$ at the same location as $\Psi_A(X)$
   - but then starts disagreeing from A in 1 *more* component than X,

   then $\Psi_A(Y)$ has 1 more computable component than $\Psi_A(X)$.
4. If we keep repeating this, eventually only computable components are left, resulting in a rank `0` point.
5. Infinitely many such rank `0` points branch off of some $\Psi_A(Z)$ (that has 1 non-computable component), giving it rank 1.
6. This way, the rank structure is built up to the top $\Psi_A(A)$, e.g. the $\Psi_A(Y)$'s witness the 1 higher rank of $\Psi_A(X)$, and so on.

The case $\alpha < \omega$

1. It remains to show the rank-faithfulness.
2. We use induction over n, the number of interleaved functionals.
3. The components of the input and the output are Turing-equivalent component-wise, that is, $\Phi_{A_i}(A_i) \equiv_T A_i$ for each i.
4. **Theorem (Owings).** $\mathrm{rk}(X \oplus Y) = \mathrm{rk}(X)$ implies $Y \leq_T X$.
5. But as A was random, all A_i are pairwise relatively random.
6. So using induction we obtain that the output rank increases by 1 with the addition of every new component A_i. $\square$

3 Proving the case $\alpha = \omega$

The challenge

1. The naïve approach (of interleaving infinitely many functionals) fails.
2. **Reason:** $\mathrm{rk}_{\Psi_A(2^\omega)}(\Psi_A(A))$ would not exist, as
   - an X has ω many components X_i in which it could differ from A;
   - so if an arbitrary finite number of components X_i has already been fixed to differ from A_i, there is always yet another component in which additional disagreement is possible;
   - this would result in yet another branching; and
   - the support of the induced measure would contain a perfect set.

The idea

1. We need a process that leads to infinite interleaving only if the input is A, but which otherwise results in finite interleaving.
2. The finite number of interleaved sequences should grow with the number of components of the input agreeing with A.

The solution: the dynamic join functional Ξ_A

1. Partition ω into the sets
   - $I_0 = \{n \in \omega : n \equiv 0 \pmod 2\}$,
   - $I_1 = \{n \in \omega : n \equiv 1 \pmod 4\}$,
   - $I_2 = \{n \in \omega : n \equiv 3 \pmod 8\}$,

   $$\vdots$$

   and let $J_0 = \omega$ and $J_{n+1} = \overline{I_0 \cup \cdots \cup I_n}$ for all $n \in \omega$.
2. The dynamic join functional Ξ_A on input X launches $\Phi_{A_0}(X_0)$ and writes its output bits into places in $J_0 = \omega$ of the output.
3. Once $\Phi_{A_0}(X_0)$ "has made progress" (= outputs a 0), it is restricted to outputting its bits into the unused places of I_0.
4. Then $\Phi_{A_1}(X_1)$ is launched in parallel, outputting its bits into so far unused locations of J_1.
5. Once both $\Phi_{A_0}(X_0)$ and $\Phi_{A_1}(X_1)$ have made progress, $\Phi_{A_1}(X_1)$ is restricted to I_1, and $\Phi_{A_2}(X_2)$ is launched writing into J_2, etc.

The solution: the dynamic join functional

1. Note how the construction ensures that all places of the output sequence become defined, independently of whether all parallel processes continue to make progress or some get stuck.
2. If $X = A$, then all processes will make always make more progress, and all countably many functionals will become launched, and all will contribute to the output sequence.
3. If $X \neq A$ then the last process that is launched is determined by what's the first component of X that disagrees with A (and by how soon this disagreement is found).

Properties of the dynamic join

1. If during the computation of $\Xi_A(X)$ the n-th process is the last that is started, then $\Xi_A(X)$ is essentially a join of $\Phi_{A_i}(X_i)$, $i \leq n$, (with the bits initially distributed in a somewhat complicated fashion).
2. We still have that the input and output of Ξ are Turing-equivalent *individually per component*, as for $\alpha < \omega$. (But *how many* components have been encoded into $\Xi(X)$ depends on X.)
3. It's easy to see that Ξ_A is total.

Proof of the case $\alpha = \omega$

1. Assume for $X \neq A$, X agrees with A in exactly n components.
2. Then $\Xi_A(X)$ agrees in exactly n components with $\Xi_A(A)$.
3. In particular, $\Xi_A(X)$ has rank n.

   (The argument is essentially the same as for $\alpha < \omega$; the finitely many "out of place" bits at the start of each component have no bearing on the infinitary notion of rank.)
4. Since now $\Xi_A(A)$ has sequences branching off it whose ranks are all $< \omega$ but converge to ω, $\Xi_A(A)$ itself must have rank ω.
5. Finally, rank-faithfulness is essentially shown as for $\alpha < \omega$.

4 Proof of the general case

Proof of the general case

1. The proof is by effective transfinite induction; some overhead is needed to handle notations for arbitrary computable ordinals.
2. Otherwise the proof is quite similar:
   - The successor case is similar to the inductive step for $\alpha < \omega$.
   - The limit case is similar to the proof for the case $\alpha = \omega$.

Conclusion

1. **Theorem, restated.** For each Δ_2^0 random degree $\mathbf{r}$ and each computable ordinal $\alpha > 0$, there is a countably supported computable measure μ and a sequence $R \in \mathbf{r}$ such that
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Thank you for your attention.

arxiv.org/abs/1707.00378