The Denjoy alternative for computable functions

Rupert Hölzl

LIAFA, CNRS & Paris 7

Joint work with Laurent Bienvenu, Joseph Miller and André Nies

Paradigm

1. In analysis, measure theory etc. there are statements that are shown to hold "almost everywhere".
2. **Intuition.** A randomly chosen real number would make the statement true.
3. Using concepts from algorithmic randomness we can try to make this more precise.

Derivatives

1 **Definition.** The *slope* of f between two reals a, b is

$$S_f(a,b) = \frac{f(a) - f(b)}{a - b}.$$

2 **Definition.**

$$\overline{D}f(z) = \limsup_{h\to 0} S_f(z, z+h) \quad \text{and} \quad \underline{D}f(z) = \liminf_{h\to 0} S_f(z, z+h)$$

(where h ranges over positive and negative numbers)

The Denjoy-Young-Saks theorem

1. **Definition.** We say that f satisfies the Denjoy alternative at x if
   - either the derivative of f at x exists
   - or $\overline{\mathrm{D}}f(x) = +\infty$ and $\underline{\mathrm{D}}f(x) = -\infty$.
2. **Intuition.** Either the derivative exists, or its existence fails in the worst possible way.
3. **The Denjoy-Young-Saks theorem.** For *any* function f and for almost every x, the Denjoy alternative for f at x holds.

Almost everywhere?

1. **Question.** Is a more precise statement than “almost everywhere” achievable?
2. In algorithmic randomness there is a hierarchy of effective notions of randomness; each giving rise to a measure 1 set.
3. They differ in the strength of the means we have to detect patterns.
4. How much effective randomness is enough to make the Denjoy-Young-Saks theorem true?

Randomness reminder

1. **Martin-Löf randomness.** Real is not random if covered by an effective sequence of c.e. sets, whose measure tends to 0 at a guaranteed minimum speed.
2. **Weak 2-randomness.** Ditto, but no speed guarantee.
3. **Computable randomness.** No computable martingale wins arbitrary amounts.

Demuth randomness

1. **Definition.** A *Demuth test* is a sequence $(\mathscr{U}_n)$ of effectively open sets such that there exists an ω-c.e. function $f : \mathbb{N} \to \mathbb{N}$ which for each n gives a c.e. index for a set of strings generating $\mathscr{U}_n$ and such that for all n, $\lambda(\mathscr{U}_n) \leq 2^{-n}$.
2. **Definition.** $x \in 2^\omega$ is called *Demuth random* if for every Demuth test $(\mathscr{U}_n)$, x belongs to only finitely many $\mathscr{U}_n$.

Difference randomness

1. **Definition.** A *difference test* is a pair $((\mathscr{U}_n)_n, \mathscr{C})$ of a uniformly c.e. sequence $(\mathscr{U}_n)_n$ of open classes and a single effectively closed class $\mathscr{C}$ such that for all n, $\lambda(\mathscr{U}_n \cap \mathscr{C}) \leq 2^{-n}$.
2. **Definition.** $x \in 2^\omega$ is called *difference random* if for any difference test $((\mathscr{U}_n)_n, \mathscr{C})$ we have $x \notin \bigcap_n (\mathscr{U}_n \cap \mathscr{C})$.

Hierarchy

1. The following inclusions hold.

$$\begin{array}{ccccccc} \mathsf{W2R} & & & & & & \\ & \searrow & & & & & \\ & & \mathsf{DiffR} & \longrightarrow & \mathsf{MLR} & \longrightarrow & \mathsf{CR} \\ & \nearrow & & & & & \\ \mathsf{DemR} & & & & & & \end{array}$$

Computable real functions

1. In order to be able to work effectively with functions and derivatives we need to make them effective.
2. **Definition.** $f : [0,1] \to \mathbb{R}$ is computable (over the reals) if its value can be effectively approximated with arbitrary precision where the input is provided as an oracle.
3. **Definition.** A real x is computable if it can be effectively approximated with arbitrary precision. Write $x \in \mathbb{R}_c$.
4. **Definition.** $f : \mathbb{R}_c \to \mathbb{R}_c$ is said to be Markov computable if from a computable Cauchy name of $x \in \mathbb{R}_c$, one can effectively compute a computable Cauchy name for $f(x)$.

Pseudoderivatives

1. Markov computable functions are not necessarily defined on all reals, so $\overline{D}f(z)$ and $\underline{D}f(z)$ may be undefined.
2. Therefore define the following *pseudoderivatives*.
3. **Definition.**

$$\underset{\sim}{D}f(z) = \liminf_{h \to 0^+} \{S_f(a,b) : a, b \in \mathrm{dom}(f) \wedge a \leq x \leq b \wedge 0 < b - a \leq h\}.$$

$$\widetilde{D}f(z) = \limsup_{h \to 0^+} \{S_f(a,b) : a, b \in \mathrm{dom}(f) \wedge a \leq x \leq b \wedge 0 < b - a \leq h\}.$$

(in the case of Markov computable f, $\mathrm{dom}(f)$ is dense)

The effectivized Denjoy alternative

1. **Definition.** Let $f :\subseteq [0,1] \to \mathbb{R}$ be partial with dense domain. f satisfies the effectivized Denjoy alternative at x if
   - either $\widetilde{D}f(x) = \underset{\sim}{D}f(x)$
   - or $\widetilde{D}f(x) = +\infty$ and $\underset{\sim}{D}f(x) = -\infty$.

Known results

1. **Theorem (Demuth).** Demuth randomness of x is enough to make the effectivized Denjoy alternative at x true for all Markov-computable functions **(notation: $x \in \mathsf{DA}$).**
2. **Definition.** A real $z \in [0,1]$ is called Denjoy random if for no Markov computable function g we have $\underset{\sim}{D}g(z) = +\infty$.
3. **Theorem (Demuth, Miller, Nies, Kučera).** T.f.a.e. for z.
   - z is Denjoy random.
   - z is computably random.
   - For every *computable* f the Denjoy alternative holds at z.

Our first main result

1. **Theorem.** Every difference random real belongs to DA.
2. **Lemma.** Difference randomness implies Π^0_1-non-porosity.
3. **Lemma.** Let x be computably random and a Π^0_1-non-porosity point. Then $x \in \mathsf{DA}$.

Our second main result

1. **Theorem.** DA is incomparable under inclusion with the set of Martin-Löf random reals.
2. **Lemma.** There is a real which is not Martin-Löf random but nonetheless satisfies the Denjoy alternative for all Markov computable functions.
3. **Lemma.** There exists a Markov computable function f for which the Denjoy alternative does not hold at Chaitin's Ω.

Lebesgue density

1. **Definition.** Lebesgue lower density of a set $\mathscr{C} \subseteq \mathbb{R}$ at a point $x \in \mathbb{R}$.

$$\rho(x|\mathscr{C}) := \liminf_{\gamma,\delta \to 0^+} \frac{\lambda([x-\gamma, x+\delta] \cap \mathscr{C})}{\lambda([x-\gamma, x+\delta])}$$

2. **Lebesgue density theorem.** Let $\mathscr{C} \subseteq \mathbb{R}$ be measurable. Then $\rho(x|\mathscr{C}) = 1$ for all $x \in \mathscr{C}$ outside a set of measure 0.

A characterization of positive density

1. **Theorem.** T.f.a.e. for $x \in \mathbb{R}$.
   - x is difference random.
   - x is Martin-Löf random and for every Π^0_1 class $\mathscr{C}$ with $x \in \mathscr{C}$ we have $\rho(x|\mathscr{C}) > 0$.
2. **Remark.** Franklin and Ng showed that this is equivalent to being Martin-Löf random and having incomplete Turing degree.
3. Very recently, Day and Miller used this result to prove that K-trivial = ML-non-cuppable.

An open question about density

1. **Remark.** It is easy to see that

$$\mathsf{W2R} \subseteq \{x \mid \forall \Pi^0_1\, \mathscr{C} : x \in \mathscr{C} \Rightarrow \rho(x|\mathscr{C}) = 1\}.$$

2. **Open question.** Is W2R equal to this set?

Thank you for your attention.