The Denjoy alternative for computable functions

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These slides *(now!)* at http://db.tt/XvLmLjBc

Dangerous cheating ahead; next eleven slides

Paradigm

1. In analysis, measure theory etc. there are statements that are shown to hold "almost everywhere".
2. **Intuition.** A randomly chosen real number would make the statement true.
3. Using concepts from algorithmic randomness we can try to make this intuition more precise.

Derivatives

1 **Definition.** The *slope* of f between two reals a, b is

$$S_f(a,b) = \frac{f(a) - f(b)}{a - b}.$$

2 **Definition.**

$$\overline{D}f(z) = \limsup_{h \to 0} S_f(z, z+h) \quad \text{and} \quad \underline{D}f(z) = \liminf_{h \to 0} S_f(z, z+h)$$

(where h ranges over positive and negative numbers)

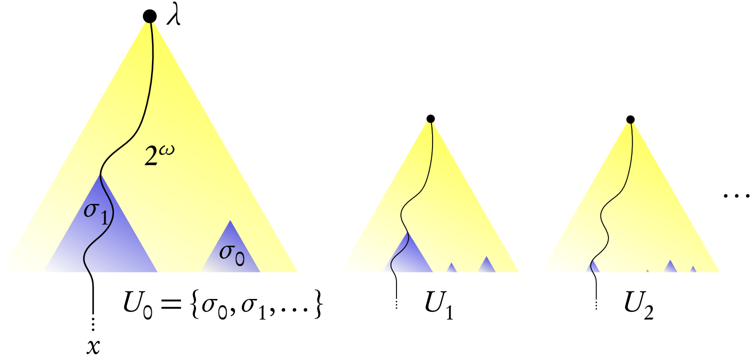
The Denjoy-Young-Saks theorem

1. **Definition.** We say that f satisfies the Denjoy alternative at x if
   - either the derivative of f at x exists
   - or $\overline{D}f(x) = +\infty$ and $\underline{D}f(x) = -\infty$.
2. **Intuition.** Either the derivative exists, or its existence fails in the worst possible way.
3. **The Denjoy-Young-Saks theorem.** For *any* function f and for almost every x, the Denjoy alternative for f at x holds.
4. **Question.** Can we use algorithmic randomness to get a more precise statement than "almost everywhere"?

Algorithmic randomness?

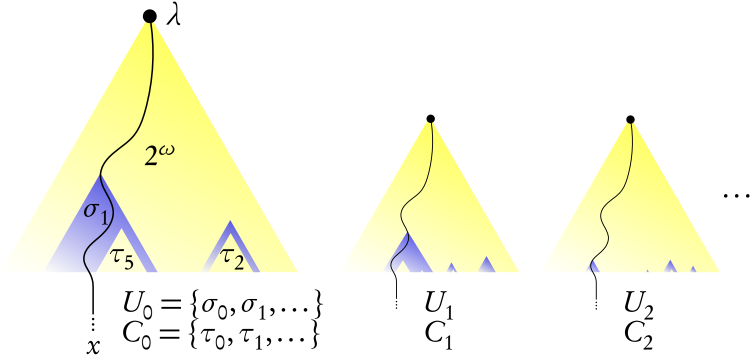
1. **Intuition.** Algorithmic randomness considers a real non-random if it is atypical, i.e., can be covered by a nullset.
2. Useless statement in this form, cf. the nullset $\{x\}$ for any x.
3. **Necessary restriction.** *Effective* nullset.
4. There is a hierarchy of effective notions of nullsets; each giving rise to a measure 1 set of randoms.

Randomness reminder



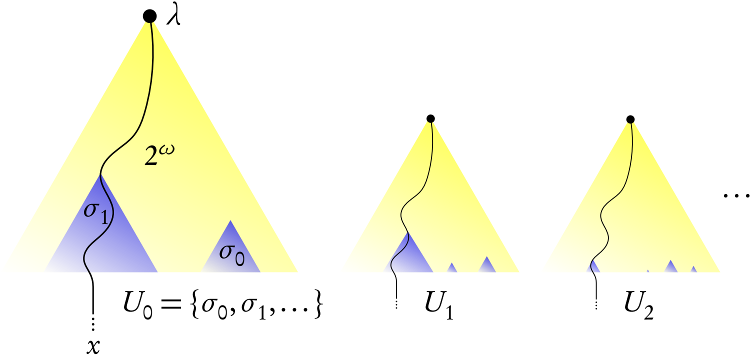
1. **Martin-Löf randomness.** Real is not random if in the intersection of an effective sequence of c.e. classes, whose measure tends to 0 at a guaranteed minimum speed.

Randomness reminder



- **Difference randomness.** Like MLR, but we are allowed to fix errors once, by uncovering a covered set again.

Randomness reminder



- **Demuth randomness.** Like MLR, but we are allowed to fix errors many times, by reseting the effective covering procedure and restarting from scratch *computably often*.

Hierarchy

1. The following inclusions hold.

$$\mathsf{DemR} \subset \mathsf{DiffR} \subset \mathsf{MLR}$$

2. How much effective randomness is enough to make the Denjoy-Young-Saks theorem true?

Computable real functions

1. In order to be able to work effectively with functions and derivatives we need to make them effective.
2. **Definition.** $f : \mathbb{R}_c \to \mathbb{R}_c$ is said to be Markov computable if from a computable Cauchy name of $x \in \mathbb{R}_c$, one can effectively compute a computable Cauchy name for $f(x)$.

A known result and a strengthening

1. **Theorem (Demuth).** Demuth randomness of x is enough to make the Denjoy alternative at x true for all Markov-computable functions **(notation: $x \in \mathsf{DA}$).**
2. **Theorem.** In fact, difference random implies DA.
3. This strengthening is significant, since Franklin and Ng showed that DiffR is equivalent to being MLR and having incomplete Turing degree.

Our second main result

1. **Theorem.** DA is not comparable under inclusion with the set of Martin-Löf random reals.
2. This is surprising since so far all known randomness notions are.

Lebesgue density

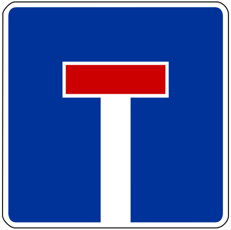
1. **Definition.** Lebesgue lower density of a set $\mathscr{C} \subseteq \mathbb{R}$ at a point $x \in \mathbb{R}$.

$$\rho(x|\mathscr{C}) := \liminf_{\gamma,\delta \to 0^+} \frac{\lambda([x-\gamma, x+\delta] \cap \mathscr{C})}{\lambda([x-\gamma, x+\delta])}$$

2. **Lebesgue density theorem.** Let $\mathscr{C} \subseteq \mathbb{R}$ be measurable. Then $\rho(x|\mathscr{C}) = 1$ for all $x \in \mathscr{C}$ outside a set of measure 0.

A characterization of positive density

1. **Theorem.** T.f.a.e. for $x \in \mathbb{R}$.
   - x is difference random.
   - x is Martin-Löf random and for every Π^0_1 class $\mathscr{C}$ with $x \in \mathscr{C}$ we have $\rho(x|\mathscr{C}) > 0$.



Thank you for your attention.