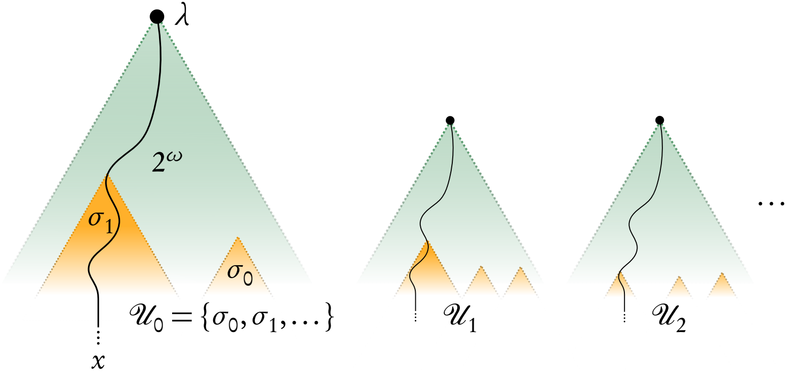
Universality, optimality, and randomness deficiency

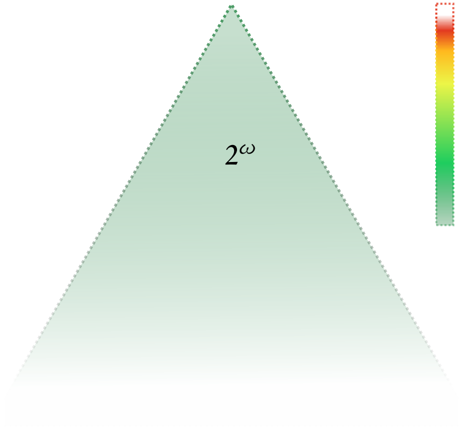
Rupert Hölzl
Universität der Bundeswehr München
Joint work with Paul Shafer

Reminder: Martin-Löf randomness



1. **Martin-Löf randomness.** Real is not random if in the intersection of an effective sequence of c.e. classes, whose measure tends to 0 at a guaranteed minimum speed.

Non-randomness heatmap



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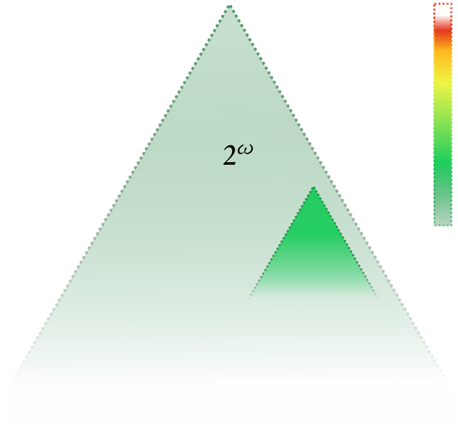

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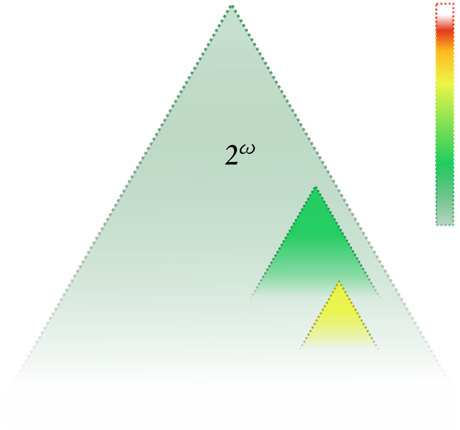

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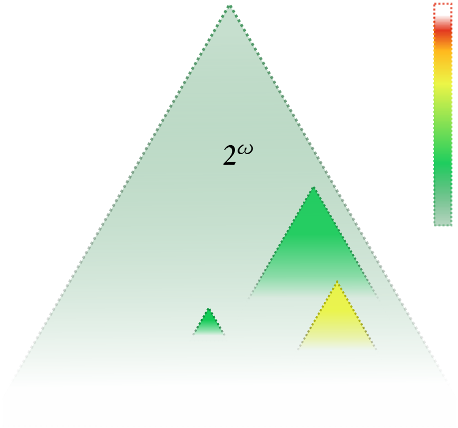

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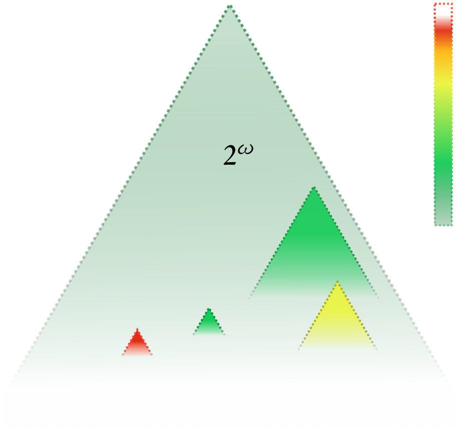

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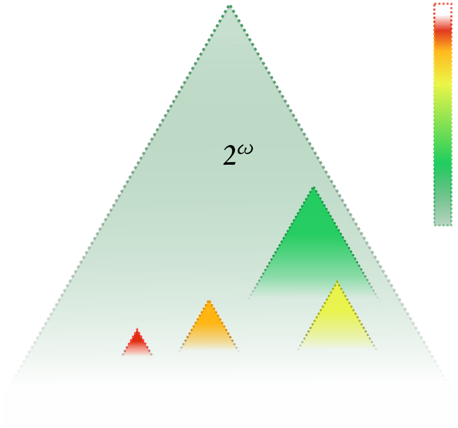

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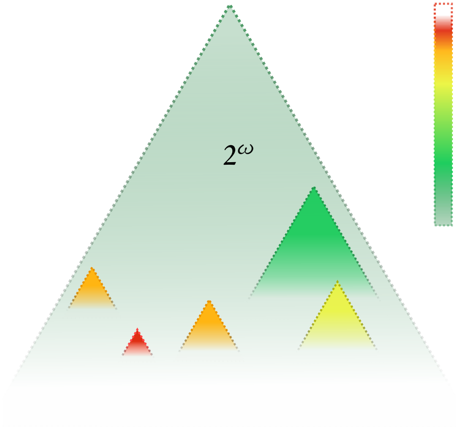

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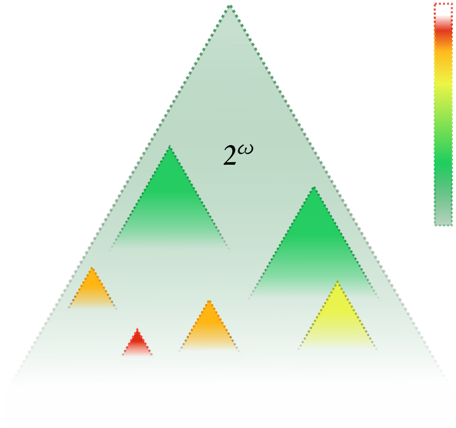

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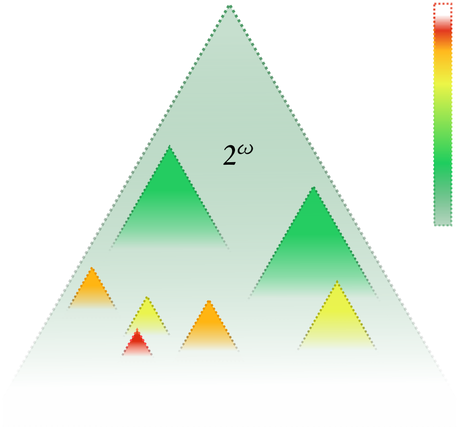

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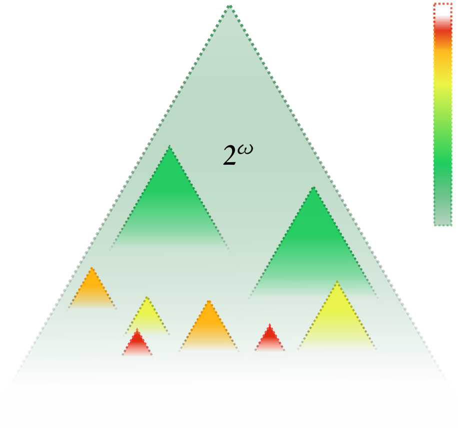

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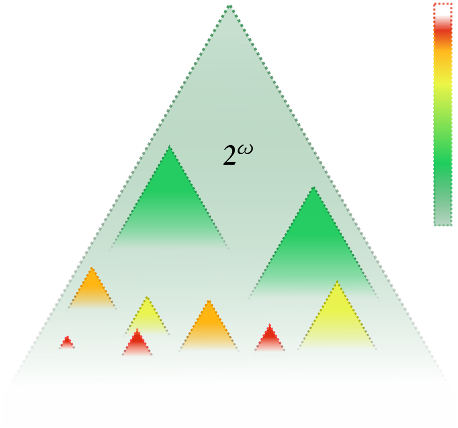

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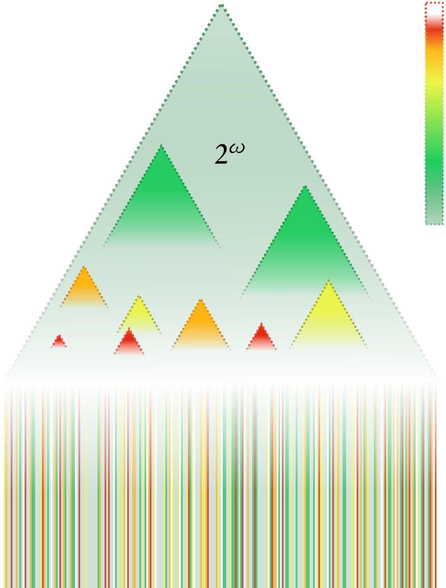

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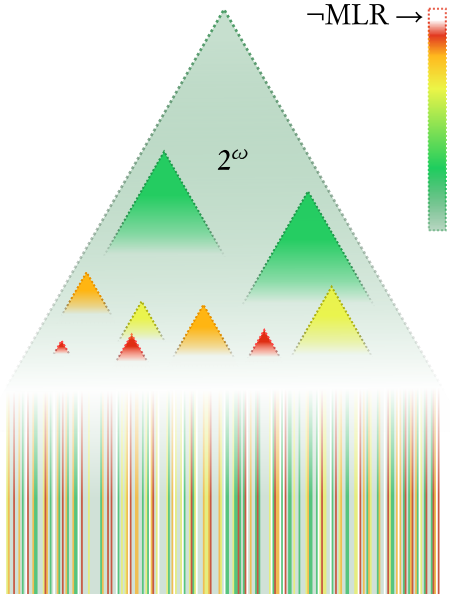


Main question

1. The picture gives a visual idea of the interna of Martin-Löf tests.
   (In general, of course, only universal tests are interesting.)
2. But there are many different universal ML-tests.
   Are their structures always similar?
   Or can they be very different?
   If yes, how different?
3. More precisely:

   How *computationally* different can they be?

Three approaches

1. We will study this question using tools of increasing precision.
   - Classical computability
   - Layerwise computability
   - Weihrauch degrees

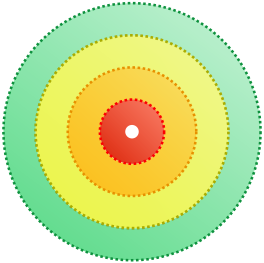
1 Preliminaries

Types of Martin-Löf tests

1. **Definition.** A Martin-Löf test $\mathscr{U}$ is *universal* if, for every Martin-Löf test $\mathscr{V}$,
$$\bigcap_{i\in\omega} \mathscr{V}_i \subseteq \bigcap_{i\in\omega} \mathscr{U}_i.$$

2. **Definition.** A Martin-Löf test $\mathscr{U}$ is *optimal* if, for every Martin-Löf test $\mathscr{V}$,
$$\exists c\, \forall i\colon\ \mathscr{V}_{i+c} \subseteq \mathscr{U}_i.$$

Caveat



1 **Definition.** A Martin-Löf test $\mathscr{V}$ is *nested* if, for all i,

$$\mathscr{V}_{i+1} \subseteq \mathscr{V}_i.$$

2 In this talk, we implicitly assume that all tests are nested, in particular when presenting intuitive arguments and proof sketches. All results also hold without that assumption.

Randomness deficiency

1. **Definition (Martin-Löf).** If $X \in \mathrm{MLR}$ and $\mathcal{U}$ is a universal ML-test, then the *randomness deficiency of X relative to $\mathcal{U}$* is defined as
$$\mathrm{rd}_{\mathcal{U}}(X) = \min\{i : X \notin \mathcal{U}_i\}.$$
2. **Intuition.** "The color of X."
(Strictly speaking "the color of X plus 1.")

2 Classical computability

Strongly non-optimal universal tests

1. **Fact.** If $\mathscr{U}$ is an ML-test such that, for every ML-test $\mathscr{V}$,

$$\forall i \exists j\colon\ \mathscr{V}_j \subseteq \mathscr{U}_i,$$

   then $\mathscr{U}$ is universal.

2. **Theorem.** The inverse is false. That is, there exist ML-tests $\mathscr{U}$ and $\mathscr{V}$ such that $\mathscr{U}$ is universal but

$$\exists i \forall j\colon\ \mathscr{V}_j \nsubseteq \mathscr{U}_i.$$

   ($\mathscr{V}$ can even be chosen optimal.)

3. **In other words.**
   - A function $f\colon \omega \to \omega$ such that $\forall i\colon\ \mathscr{V}_{f(i)} \subseteq \mathscr{U}_i$ need not exist.
   - That is, there are strongly non-optimal universal tests.

Non-optimal tests and incomputability

1. **Theorem.** There are ML-tests $\mathscr{W}$ and $\mathscr{V}$ such that
   - $\mathscr{W}$ is universal,
   - $\forall i \exists j\colon \mathscr{V}_j \subseteq \mathscr{W}_i$,
   - if $f\colon \omega \to \omega$ satisfies $\forall i\colon \mathscr{V}_{f(i)} \subseteq \mathscr{W}_i$, then $f \geq_{\mathrm{T}} 0''$.

   ($\mathscr{V}$ can even be chosen optimal.)
2. **Summary.**
   - There are strongly non-optimal universal tests.
   - Even when functions $f\colon \omega \to \omega$ such that $\forall i\colon \mathscr{V}_{f(i)} \subseteq \mathscr{U}_i$ exist, they may be strongly incomputable.

3 Layerwise computability

Layerwise computability

1. Let $\mathscr{U}$ be a universal ML-test.
2. **Definition.** A function f is *exactly $\mathscr{U}$-layerwise computable* if there is a Turing functional Φ such that

$$\forall X \in \mathrm{MLR}\colon\ \Phi(\langle X, \mathrm{rd}_{\mathscr{U}}(X)\rangle) = f(X).$$

3. **Intuition.** "If you know the color of X you can compute $f(X)$."
4. **Definition (Hoyrup and Rojas).** A function f is *$\mathscr{U}$-layerwise computable* if there is a Turing functional Φ such that

$$\forall i\, \forall X \in 2^{\omega} \setminus \mathscr{U}_i\colon\ \Phi(\langle X, i\rangle) = f(X).$$

5. **Different tests give different colors to X.**
How does this influence the computability of $f(X)$?

Layerwise computability depends on the test

1. **Theorem (Hoyrup and Rojas).** Let $\mathscr{U}$ be an *optimal* ML-test, and let $\mathscr{A} \subseteq 2^\omega$. Then $\mathscr{A}$ is effectively measurable if and only if $\mathbb{1}_{\mathscr{A}}$ is $\mathscr{U}$-layerwise computable.
2. **Theorem.** There is an effectively measurable set $\mathscr{A} \subseteq 2^\omega$ and a universal ML-test $\mathscr{W}$ such that $\mathbb{1}_{\mathscr{A}}$ is not $\mathscr{W}$-layerwise computable.
3. **Corollary.** Layerwise computability depends on the test.

   (That is, there are universal ML-tests $\mathscr{U}$ and $\mathscr{W}$ and a function that is $\mathscr{U}$-layerwise computable but not $\mathscr{W}$-layerwise computable.)

Layerwise computable vs. *exactly* layerwise computable

1. **Theorem.** Let $\mathscr{U}$ be a universal ML-test. Then $\mathrm{rd}_{\mathscr{U}}$ is not $\mathscr{U}$-layerwise computable.
2. **Corollary.** For every universal ML-test $\mathscr{U}$ there is a function that is exactly $\mathscr{U}$-layerwise computable but not $\mathscr{U}$-layerwise computable.
3. **Summary.** The class of exactly layerwise computable functions is *always* strictly larger than the class of layerwise computable functions.

Exact layerwise computability depends on the test

1 **Theorem.** Exact layerwise computability depends on the test.

(That is, there are universal ML-tests $\mathscr{V}$ and $\mathscr{W}$ and a function that is exactly $\mathscr{W}$-layerwise computable but not exactly $\mathscr{V}$-layerwise computable.)

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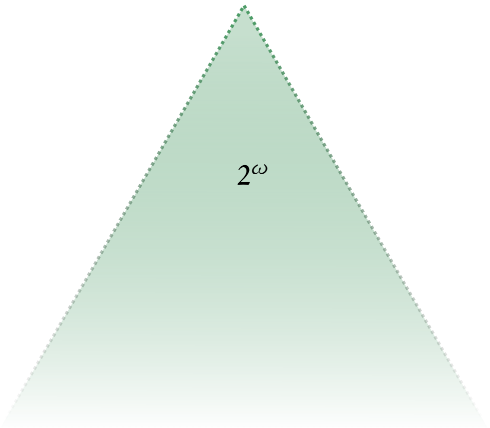

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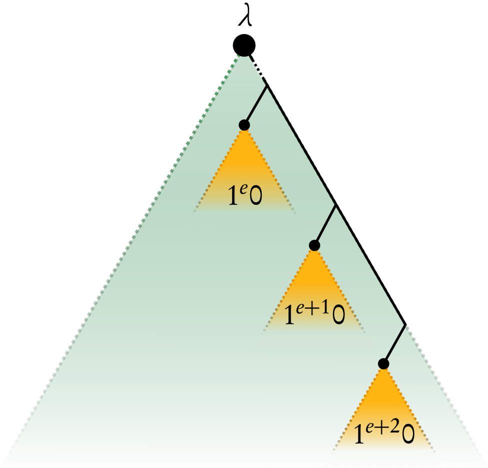

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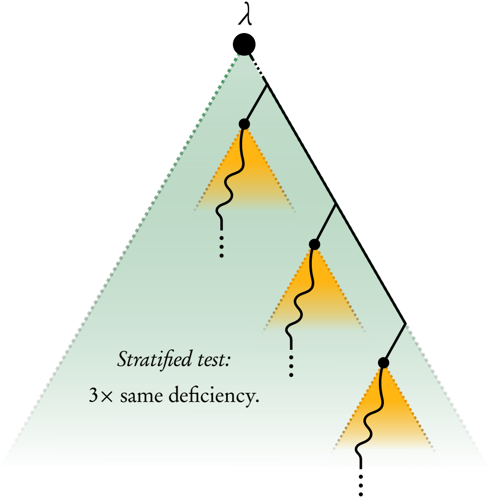

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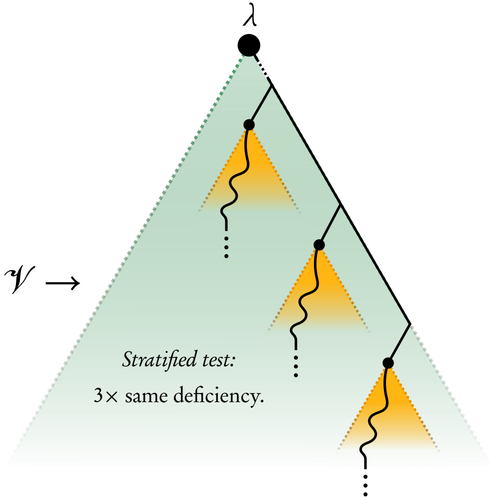

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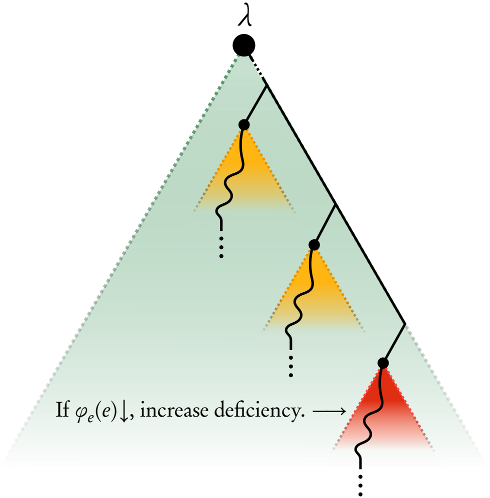

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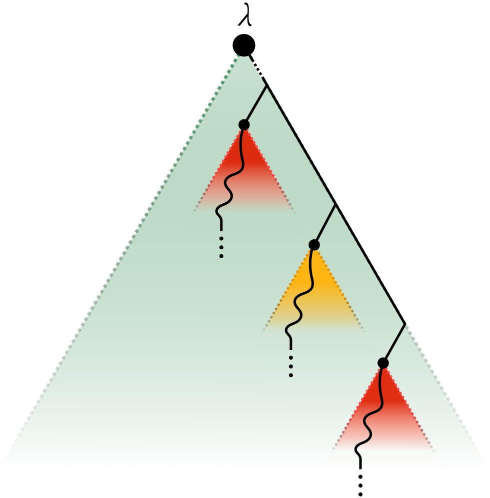

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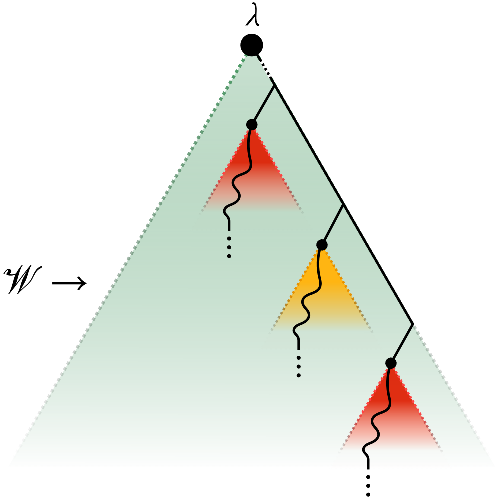

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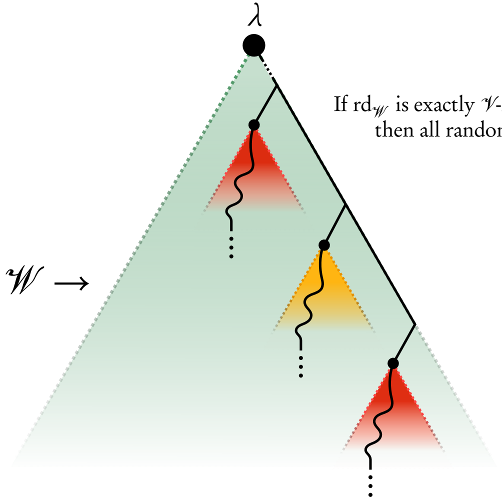

If $\mathrm{rd}_{\mathscr{W}}$ is exactly $\mathscr{V}$-layerwise computable, then all randoms compute $\emptyset'$. ↯

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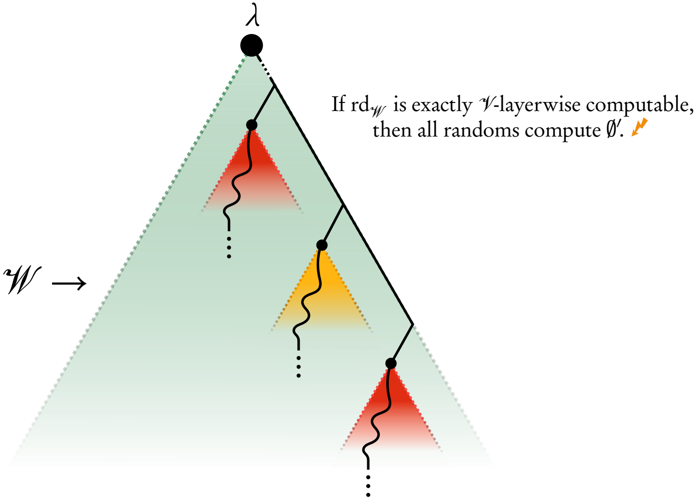

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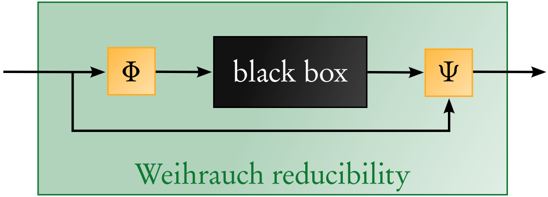
4 Weihrauch degrees

Reminder: Weihrauch degrees

1. **General idea.** We look at mathematical tasks where a problem is given to a black box, and the black box has to give back a solution to the problem.
2. **Example.** Given a bounded sequence of rationals, the black box has to produce an accumulation point.
3. **Reducibility.** Assume we have a black box solving a certain problem $\mathscr{A}$. Can we use it to solve another problem $\mathscr{B}$?
4. **Approach.**
   - Code an instance B of problem $\mathscr{B}$ into a valid instance A of problem $\mathscr{A}$.
   - Run the black box for $\mathscr{A}$ on A.
   - Convert the solution for A back into a solution for B.

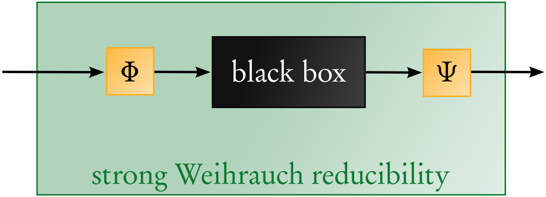
Reminder: Weihrauch reducibilities



1. Φ and Ψ are both Turing functionals.
2. **Weak reducibility.** The decoding procedure Ψ has access to the original input.
3. That is, if B is the black box, the computed function is

$$x \mapsto \Psi(\langle B(\Phi(x)), x \rangle).$$

Reminder: Weihrauch reducibilities



1. Φ and Ψ are both Turing functionals.
2. **Strong reducibility.** The decoding procedure Ψ *does not* have access to the original input.
3. That is, if B is the black box, the computed function is

$$x \mapsto \Psi(\mathrm{B}(\Phi(x))).$$

The principles LAY and RD

1. Let $\mathscr{U}$ be a universal ML-test.
2. **Definition (Brattka, Gherardi, Hölzl).**

$$\begin{array}{rccl} \mathrm{LAY}_{\mathscr{U}}\colon & \mathrm{MLR} & \rightrightarrows & \omega \\ & X & \longmapsto & \{i\colon X \notin \mathscr{U}_i\}. \end{array}$$

3. **Definition.** $\begin{array}{rccl} \mathrm{RD}_{\mathscr{U}}\colon & \mathrm{MLR} & \longrightarrow & \omega \\ & X & \longmapsto & \mathrm{rd}_{\mathscr{U}}(X). \end{array}$

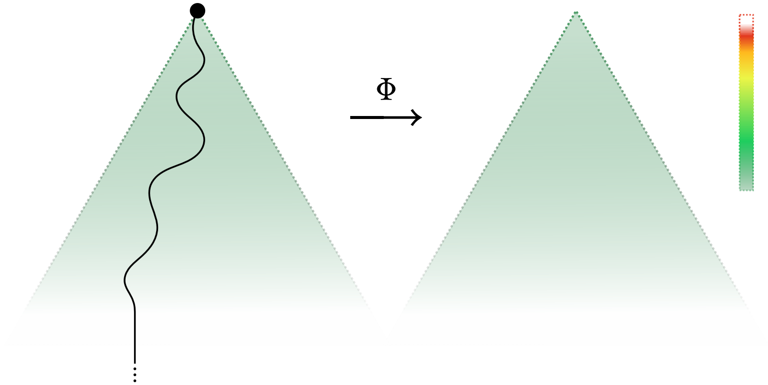
   (We use $\mathrm{RD}_{\mathscr{U}}$ for the principle, and $\mathrm{rd}_{\mathscr{U}}$ for the property of sequences.)
4. **Intuition.** "For a given sequence X, what is its color?"

Strong equivalence of different LAY's

1 **Proposition.** If $\mathscr{U}$ and $\mathscr{V}$ are universal ML-tests, then

$$\mathrm{LAY}_{\mathscr{V}} \leq_{\mathrm{sW}} \mathrm{LAY}_{\mathscr{U}}.$$

Strong equivalence of different LAY's



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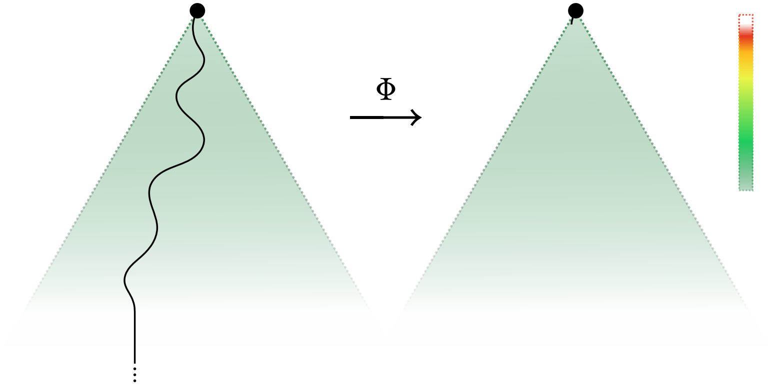

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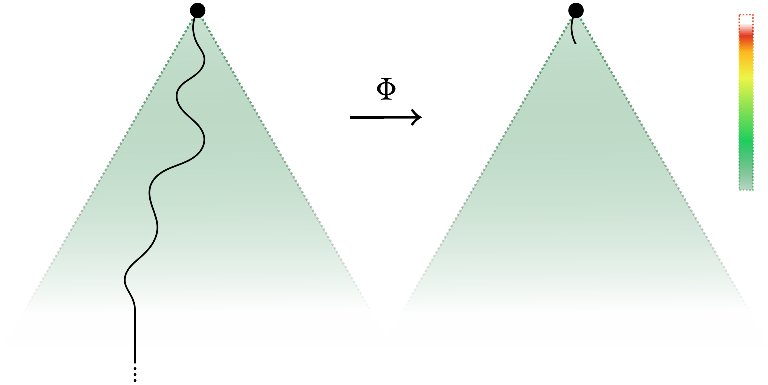

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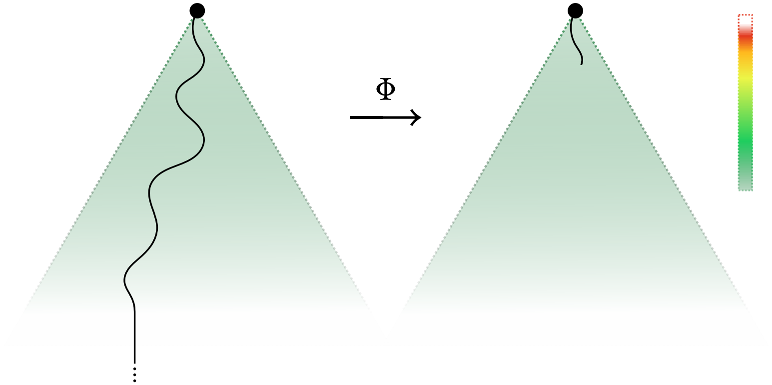

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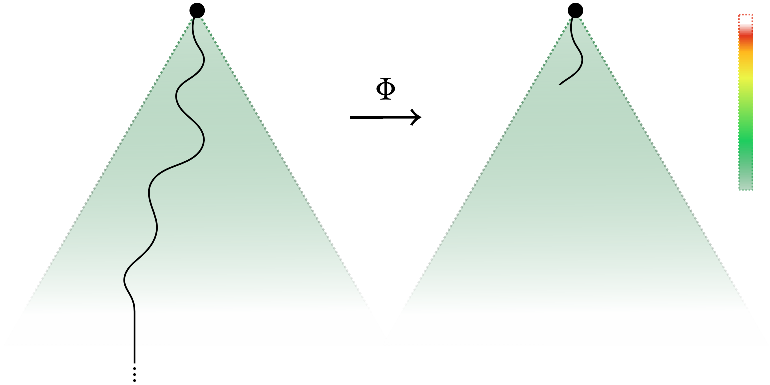

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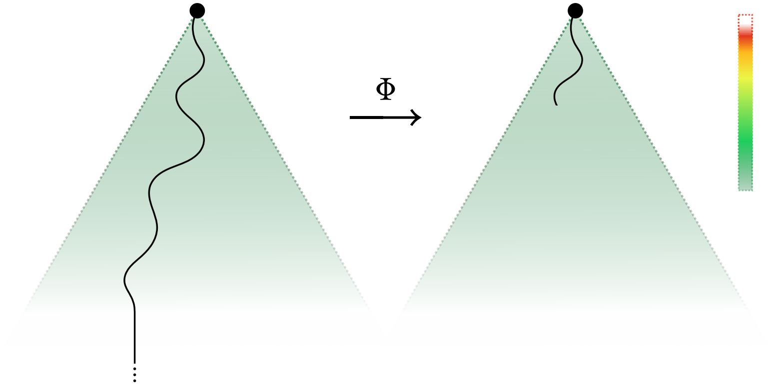

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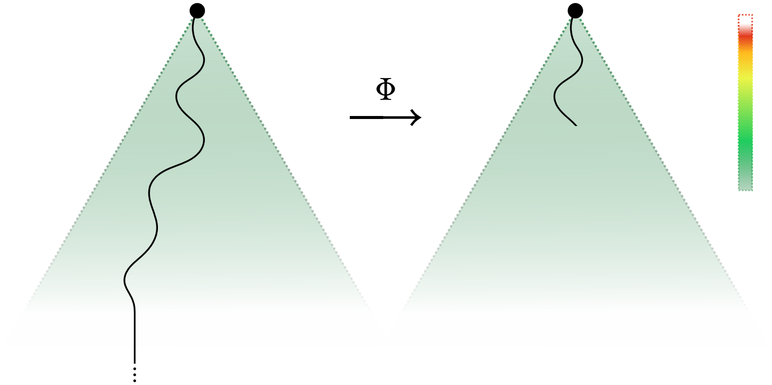

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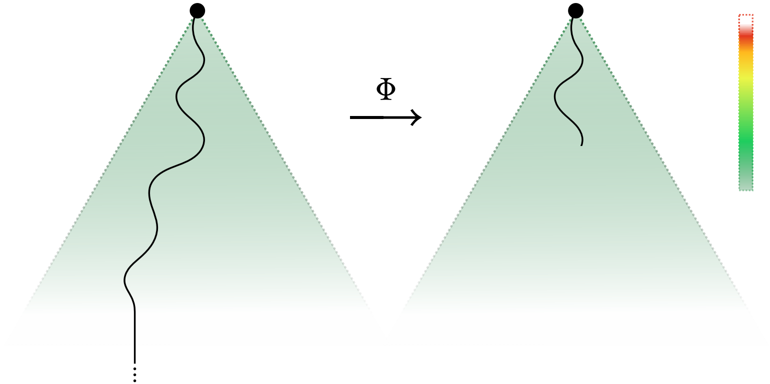

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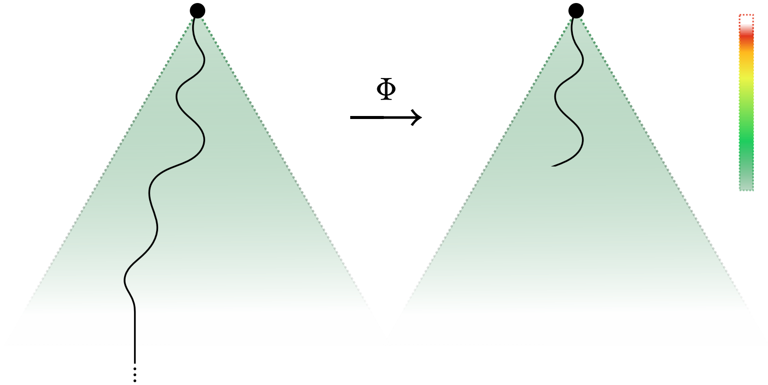

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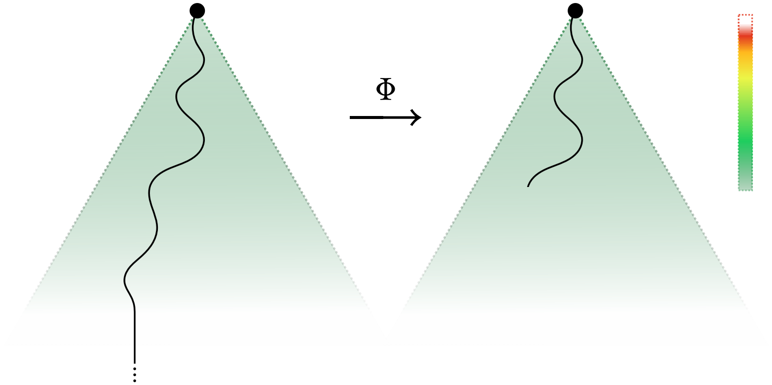

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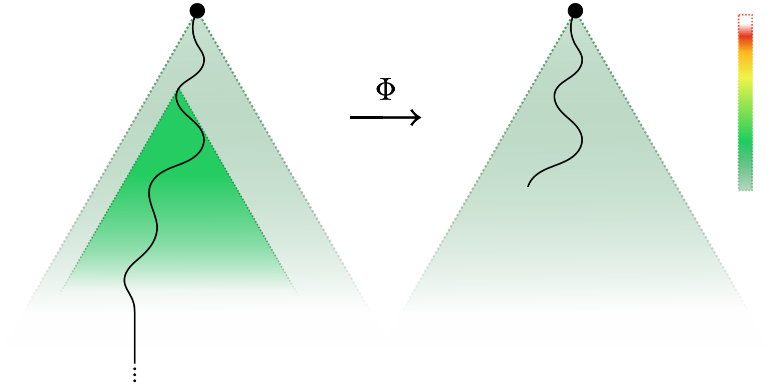

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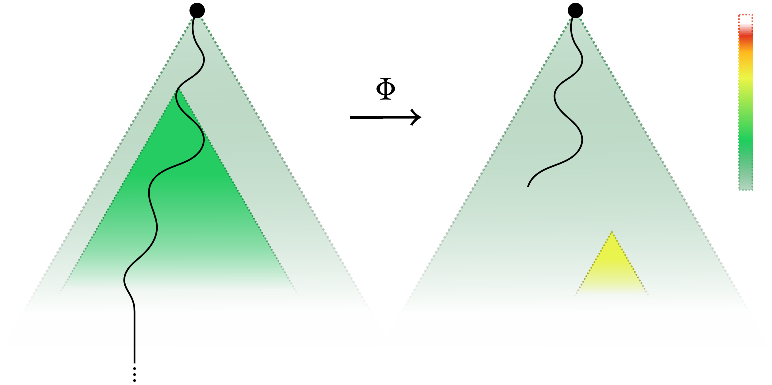

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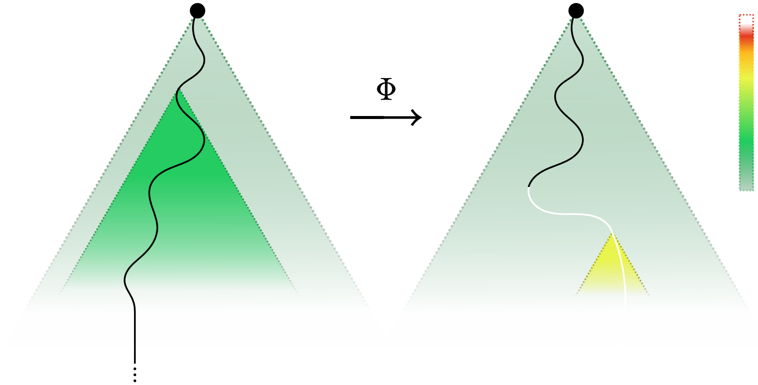

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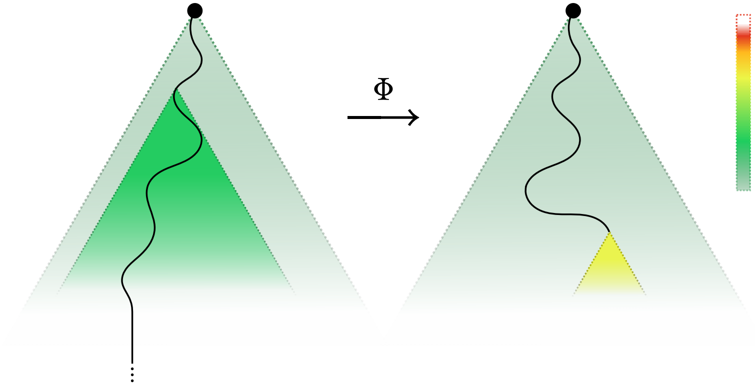

1 **Proposition.** If $\mathscr{U}$ and $\mathscr{V}$ are universal ML-tests, then

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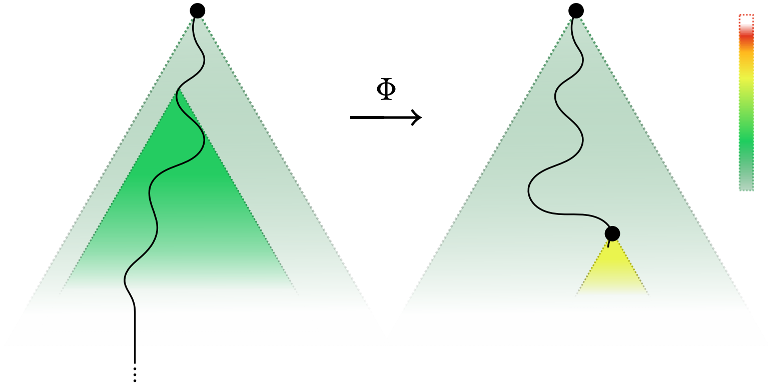

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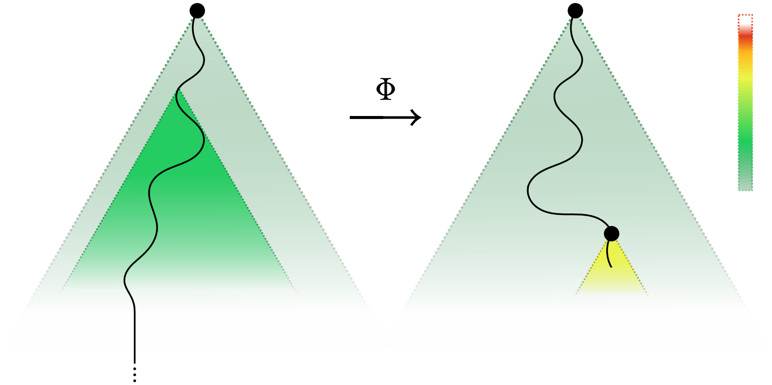

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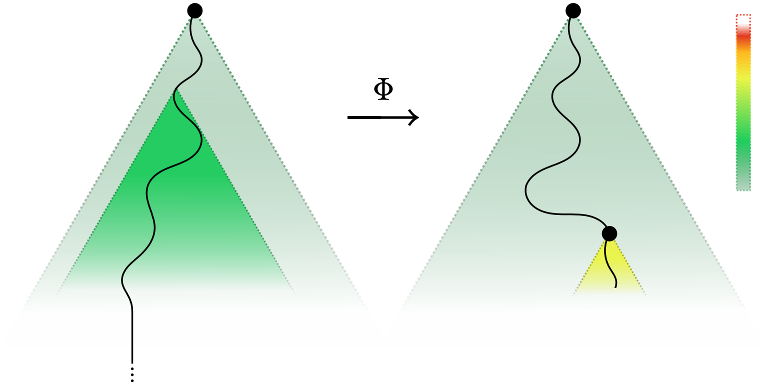

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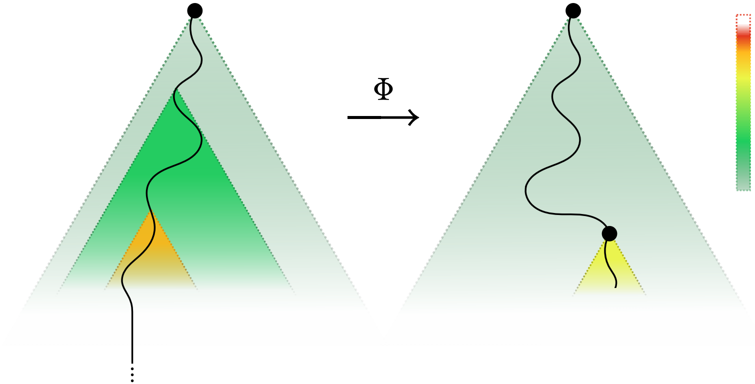

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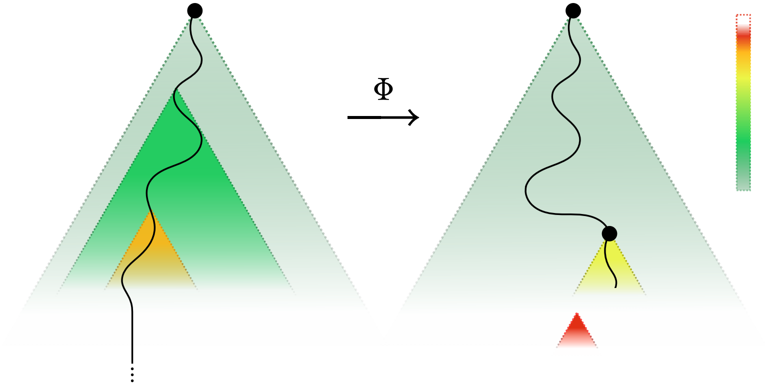

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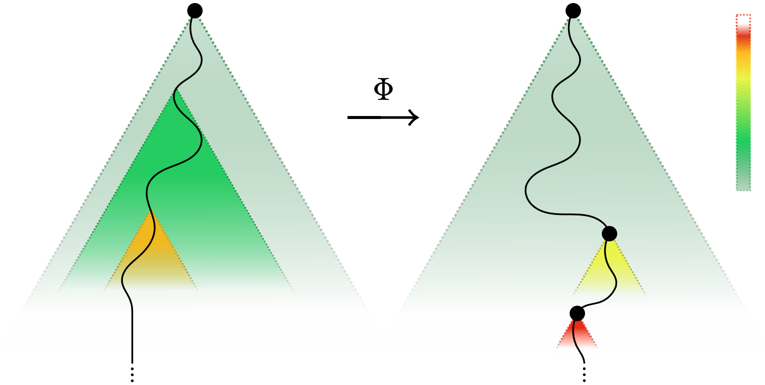

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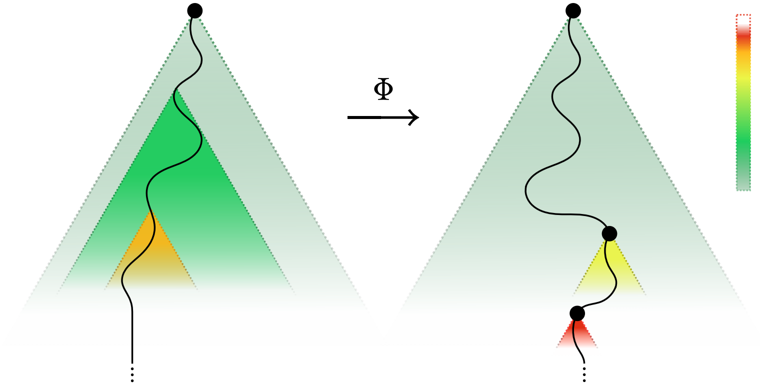

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Weak equivalence of LAY and RD

1. **Theorem.** Let $\mathscr{U}$ and $\mathscr{V}$ be universal ML-tests. Then

$$\mathrm{RD}_{\mathscr{V}} \equiv_{\mathrm{W}} \mathrm{LAY}_{\mathscr{U}}.$$

Proof. Same idea as for the previous proof, but more involved.

2. **Proposition.** Let $\mathscr{U}$ and $\mathscr{V}$ be universal ML-tests. Then

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1 **Theorem.** Let $\mathscr{U}$ and $\mathscr{V}$ be universal ML-tests. Then

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Let $n \geq \max(\mathrm{rd}_{\mathscr{U}}(\Phi(X)), \mathrm{rd}_{\mathscr{U}}(\Phi(Y)))$.
Then $\mathrm{rd}_{\mathscr{V}}(X) = \Psi(n) = \mathrm{rd}_{\mathscr{V}}(Y)$. ↯ □

Open question & literary references

1 **Conjecture.** There exist universal tests $\mathscr{U}$ and $\mathscr{V}$ such that

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Thank you for your attention.