Universality, optimality, and randomness deficiency

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These slides (now!) at hoelzl.fr/uord
Reminder: Martin-Löf randomness

1 Martin-Löf randomness. Real is not random if in the intersection of an effective sequence of c.e. classes, whose measure tends to 0 at a guaranteed minimum speed.
Non-randomness heatmap

$2^\omega$
Non-randomness heatmap

\[ 2^\omega \]
Non-randomness heatmap

$2^\omega$
Non-randomness heatmap
Non-randomness heatmap

$2^{\omega}$
Non-randomness heatmap
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Non-randomness heatmap
Non-randomness heatmap

$2^\omega$
Non-randomness heatmap
Non-randomness heatmap
Main question

1. The picture gives a visual idea of the interna of Martin-Löf tests. (In general, of course, only universal tests are interesting.)

2. But there are many different universal ML-tests. Are their structures always similar? Or can they be very different? If yes, how different?

3. More precisely:

   How *computationally* different can they be?
Three approaches

1. We will study this question using tools of increasing precision.
   - Classical computability
   - Layerwise computability
   - Weihrauch degrees
1 Preliminaries
1 **Definition.** A Martin-Löf test $\mathcal{U}$ is *universal* if, for every Martin-Löf test $\mathcal{V}$,

$$\bigcap_{i \in \omega} \mathcal{V}_i \subseteq \bigcap_{i \in \omega} \mathcal{U}_i.$$ 

2 **Definition.** A Martin-Löf test $\mathcal{U}$ is *optimal* if, for every Martin-Löf test $\mathcal{V}$,

$$\exists c \forall i: \mathcal{V}_{i+c} \subseteq \mathcal{U}_i.$$
**Caveat**

1. **Definition.** A Martin-Löf test $\mathcal{V}$ is *nested* if, for all $i$,

$$\mathcal{V}_{i+1} \subseteq \mathcal{V}_i.$$  

2. In this talk, we implicitly assume that all tests are nested, in particular when presenting intuitive arguments and proof sketches. All results also hold without that assumption.
Definition (Martin-Löf). If $X \in \text{MLR}$ and $\mathcal{U}$ is a universal ML-test, then the \textit{randomness deficiency of $X$ relative to $\mathcal{U}$} is defined as

$$\text{rd}_{\mathcal{U}}(X) = \min\{i : X \notin \mathcal{U}_i\}.$$ 

Intuition. “The color of $X$.”

(Strictly speaking “the color of $X$ plus 1.”)
Classical computability
Strongly non-optimal universal tests

1 **Fact.** If $\mathcal{U}$ is an ML-test such that, for every ML-test $\mathcal{V}$,

$$\forall i \exists j : \mathcal{V}_j \subseteq \mathcal{U}_i,$$

then $\mathcal{U}$ is universal.

2 **Theorem.** The inverse is false. That is, there exist ML-tests $\mathcal{U}$ and $\mathcal{V}$ such that $\mathcal{U}$ is universal but

$$\exists i \forall j : \mathcal{V}_j \not\subseteq \mathcal{U}_i.$$

($\mathcal{V}$ can even be chosen optimal.)

3 **In other words.**

- A function $f : \omega \to \omega$ such that $\forall i : \mathcal{V}_{f(i)} \subseteq \mathcal{U}_i$ need not exist.
- That is, there are strongly non-optimal universal tests.
1 **Theorem.** There are ML-tests $\mathcal{W}$ and $\mathcal{V}$ such that
- $\mathcal{W}$ is universal,
- $\forall i \exists j: \mathcal{V}_j \subseteq \mathcal{W}_i$,
- if $f: \omega \rightarrow \omega$ satisfies $\forall i: \mathcal{V}_{f(i)} \subseteq \mathcal{W}_i$, then $f \geq_T 0''$.

($\mathcal{V}$ can even be chosen optimal.)

2 **Summary.**
- There are strongly non-optimal universal tests.
- Even when functions $f: \omega \rightarrow \omega$ such that $\forall i: \mathcal{V}_{f(i)} \subseteq \mathcal{U}_i$ exist, they may be strongly incomputable.
Layerwise computability
Layerwise computability

1. Let $U$ be a universal ML-test.

2. **Definition.** A function $f$ is exactly $U$-layerwise computable if there is a Turing functional $\Phi$ such that

$$\forall X \in \text{MLR}: \Phi(\langle X, \text{rd}_U(X) \rangle) = f(X).$$

3. **Intuition.** “If you know the color of $X$ you can compute $f(X)$.”

4. **Definition (Hoyrup and Rojas).** A function $f$ is $U$-layerwise computable if there is a Turing functional $\Phi$ such that

$$\forall i \forall X \in 2^\omega \setminus U_i: \Phi(\langle X, i \rangle) = f(X).$$

5. **Different tests give different colors to $X$.** How does this influence the computability of $f(X)$?
Layerwise computability depends on the test

1 Theorem (Hoyrup and Rojas). Let $\mathcal{U}$ be an optimal ML-test, and let $\mathcal{A} \subseteq 2^\omega$. Then $\mathcal{A}$ is effectively measurable if and only if $\mathbb{1}_\mathcal{A}$ is $\mathcal{U}$-layerwise computable.

2 Theorem. There is an effectively measurable set $\mathcal{A} \subseteq 2^\omega$ and a universal ML-test $\mathcal{W}$ such that $\mathbb{1}_\mathcal{A}$ is not $\mathcal{W}$-layerwise computable.

Proof idea. Ensure that no $\Phi_i$ $\mathcal{W}$-layerwise computes $\mathbb{1}_\mathcal{A}$:

 Wait for $\Phi_i(\langle \sigma, i \rangle)$ to converge to either 0 or 1 on some $[\sigma]$. In the first case put $[\sigma]$ into $\mathcal{A}$; in the second case ensure that $[\sigma]$ is never put into $\mathcal{A}$. Then $[\sigma]$ witnesses the failure of $\Phi_i$. □

3 Corollary. Layerwise computability depends on the test.

(That is, there are universal ML-tests $\mathcal{U}$ and $\mathcal{W}$ and a function that is $\mathcal{U}$-layerwise computable but not $\mathcal{W}$-layerwise computable.)
Layerwise computable vs. \textit{exactly} layerwise computable

1. \textbf{Theorem.} Let $\mathcal{U}$ be a universal ML-test. Then $\text{rd}_{\mathcal{U}}$ is not $\mathcal{U}$-layerwise computable.

2. \textbf{Corollary.} For every universal ML-test $\mathcal{U}$ there is a function that is exactly $\mathcal{U}$-layerwise computable but not $\mathcal{U}$-layerwise computable.

3. \textbf{Summary.} The class of exactly layerwise computable functions is \textit{always} strictly larger than the class of layerwise computable functions.
Theorem. Exact layerwise computability depends on the test. (That is, there are universal ML-tests $\mathcal{V}$ and $\mathcal{W}$ and a function that is exactly $\mathcal{W}$-layerwise computable but not exactly $\mathcal{V}$-layerwise computable.)
1 Theorem. Exact layerwise computability depends on the test. (That is, there are universal ML-tests $V$ and $W$ and a function that is exactly $W$-layerwise computable but not exactly $V$-layerwise computable.)
Theorem. Exact layerwise computability depends on the test.

(That is, there are universal ML-tests $\mathcal{V}$ and $\mathcal{W}$ and a function that is exactly $\mathcal{W}$-layerwise computable but not exactly $\mathcal{V}$-layerwise computable.)
1 **Theorem.** Exact layerwise computability depends on the test.
(That is, there are universal ML-tests $\mathcal{V}$ and $\mathcal{W}$ and a function that is exactly $\mathcal{W}$-layerwise computable but not exactly $\mathcal{V}$-layerwise computable.)
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Theorem. Exact layerwise computability depends on the test.
(That is, there are universal ML-tests \( \mathcal{V} \) and \( \mathcal{W} \) and a function that is exactly \( \mathcal{W} \)-layerwise computable but not exactly \( \mathcal{V} \)-layerwise computable.)
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(That is, there are universal ML-tests $\mathcal{V}$ and $\mathcal{W}$ and a function that is exactly $\mathcal{W}$-layerwise computable but not exactly $\mathcal{V}$-layerwise computable.)
1 Theorem. Exact layerwise computability depends on the test.
(That is, there are universal ML-tests $\forall$ and $\forall'$ and a function that is exactly $\forall'$-layerwise computable but not exactly $\forall$-layerwise computable.)
4

Weihrauch degrees
1 General idea. We look at mathematical tasks where a problem is given to a black box, and the black box has to give back a solution to the problem.

2 Example. Given a bounded sequence of rationals, the black box has to produce an accumulation point.

3 Reducibility. Assume we have a black box solving a certain problem $\mathcal{A}$. Can we use it to solve another problem $\mathcal{B}$?

4 Approach.
   - Code an instance $B$ of problem $\mathcal{B}$ into a valid instance $A$ of problem $\mathcal{A}$.
   - Run the black box for $\mathcal{A}$ on $A$.
   - Convert the solution for $A$ back into a solution for $B$. 
Reminder: Weihrauch reducibilities

1. $\Phi$ and $\Psi$ are both Turing functionals.
2. **Weak reducibility.** The decoding procedure $\Psi$ has access to the original input.
3. That is, if $B$ is the black box, the computed function is

\[ x \mapsto \Psi(B(\Phi(x)), x). \]
Reminder: Weihrauch reducibilities

1. $\Phi$ and $\Psi$ are both Turing functionals.
2. **Strong reducibility.** The decoding procedure $\Psi$ *does not* have access to the original input.
3. That is, if $B$ is the black box, the computed function is

$$x \mapsto \Psi(B(\Phi(x))).$$
1 Let $\mathcal{U}$ be a universal ML-test.

2 **Definition (Brattka, Gherardi, Hölzl).**

   \[
   \text{LAY}_\mathcal{U} : \quad \text{MLR} \quad \implies \quad \omega \\
   X \quad \iff \quad \{i : X \notin \mathcal{U}_i\}.
   \]

3 **Definition.**  \[
   \text{RD}_\mathcal{U} : \quad \text{MLR} \quad \implies \quad \omega \\
   X \quad \iff \quad \text{rd}_\mathcal{U}(X).
   \]

   (We use $\text{RD}_\mathcal{U}$ for the principle, and $\text{rd}_\mathcal{U}$ for the property of sequences.)

4 **Intuition.** “For a given sequence $X$, what is its color?”
Proposition. If $\mathcal{U}$ and $\mathcal{V}$ are universal ML-tests, then

$$\text{LAY}_\mathcal{V} \leq_{sW} \text{LAY}_\mathcal{U}.$$
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Proposition. If $\mathcal{U}$ and $\mathcal{V}$ are universal ML-tests, then

$$\text{LAY}_\mathcal{V} \leq_{\text{sw}} \text{LAY}_\mathcal{U}.$$
Proposition. If $\mathcal{U}$ and $\mathcal{V}$ are universal ML-tests, then

$$\text{LAY}_\mathcal{V} \leq_{SW} \text{LAY}_\mathcal{U}.$$
Strong equivalence of different LAY’s

1 Proposition. If $\mathcal{U}$ and $\mathcal{V}$ are universal ML-tests, then

\[ \text{LAY}_{\mathcal{V}} \leq_{sW} \text{LAY}_{\mathcal{U}}. \]
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Strong equivalence of different LAY’s

1 Proposition. If \( U \) and \( V \) are universal ML-tests, then

\[
\text{LAY}_V \leq_{SW} \text{LAY}_U.
\]
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\text{LAY}_{\mathcal{V}} \leq_{sW} \text{LAY}_{\mathcal{U}}.
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Proposition. If $\mathcal{U}$ and $\mathcal{V}$ are universal ML-tests, then

$$\text{LAY}_\mathcal{V} \leq_{SW} \text{LAY}_\mathcal{U}.$$
Weak equivalence of LAY and RD

1. **Theorem.** Let $\mathcal{U}$ and $\mathcal{V}$ be universal ML-tests. Then

$$\text{RD}_\mathcal{V} \equiv_\mathcal{W} \text{LAY}_\mathcal{U}.$$ 

**Proof.** Same idea as for the previous proof, but more involved.

2. **Proposition.** Let $\mathcal{U}$ and $\mathcal{V}$ be universal ML-tests. Then

$$\text{RD}_\mathcal{V} \not\leq_{\mathcal{SW}} \text{LAY}_\mathcal{U}.$$
Weak equivalence of LAY and RD

1 Theorem. Let $\mathcal{U}$ and $\mathcal{V}$ be universal ML-tests. Then

$$\text{RD}_\mathcal{V} \equiv_W \text{LAY}_\mathcal{U}.$$ 

Proof. Same idea as for the previous proof, but more involved.

2 Proposition. Let $\mathcal{U}$ and $\mathcal{V}$ be universal ML-tests. Then

$$\text{RD}_\mathcal{V} \not\leq_W \text{LAY}_\mathcal{U}.$$ 

Proof. Suppose $\text{RD}_\mathcal{V} \leq_W \text{LAY}_\mathcal{U}$ witnessed by $\Phi$ and $\Psi$. 
Weak equivalence of LAY and RD

1. **Theorem.** Let $\mathcal{U}$ and $\mathcal{V}$ be universal ML-tests. Then

$$RD_{\mathcal{V}} \equiv_{W} LAY_{\mathcal{U}}.$$ 

**Proof.** Same idea as for the previous proof, but more involved.

2. **Proposition.** Let $\mathcal{U}$ and $\mathcal{V}$ be universal ML-tests. Then

$$RD_{\mathcal{V}} \not\leq_{sW} LAY_{\mathcal{U}}.$$

**Proof.** Suppose $RD_{\mathcal{V}} \leq_{sW} LAY_{\mathcal{U}}$ witnessed by $\Phi$ and $\Psi$. Then $X \in \text{MLR}$ and $n \geq \text{rd}_{\mathcal{U}}(\Phi(X))$ imply $\Psi(n) = \text{rd}_{\mathcal{V}}(X)$. 
1 Theorem. Let $\mathcal{U}$ and $\mathcal{V}$ be universal ML-tests. Then

$$\text{RD}_{\mathcal{V}} \equiv_{W} \text{LAY}_{\mathcal{U}}.$$  

Proof. Same idea as for the previous proof, but more involved.

2 Proposition. Let $\mathcal{U}$ and $\mathcal{V}$ be universal ML-tests. Then

$$\text{RD}_{\mathcal{V}} \not\leq_{sW} \text{LAY}_{\mathcal{U}}.$$  

Proof. Suppose $\text{RD}_{\mathcal{V}} \leq_{sW} \text{LAY}_{\mathcal{U}}$ witnessed by $\Phi$ and $\Psi$. Then $X \in \text{MLR}$ and $n \geq \text{rd}_{\mathcal{U}}(\Phi(X))$ imply $\Psi(n) = \text{rd}_{\mathcal{V}}(X)$. Fix $X, Y \in \text{MLR}$ with $\text{rd}_{\mathcal{V}}(X) \neq \text{rd}_{\mathcal{V}}(Y)$. 


Weak equivalence of LAY and RD

1. **Theorem.** Let $\mathcal{U}$ and $\mathcal{V}$ be universal ML-tests. Then

$$\text{RD}_\mathcal{V} \equiv_{\text{W}} \text{LAY}_\mathcal{U}.$$  

**Proof.** Same idea as for the previous proof, but more involved.

2. **Proposition.** Let $\mathcal{U}$ and $\mathcal{V}$ be universal ML-tests. Then

$$\text{RD}_\mathcal{V} \not<_{\text{SW}} \text{LAY}_\mathcal{U}.$$  

**Proof.** Suppose $\text{RD}_\mathcal{V} \leq_{\text{SW}} \text{LAY}_\mathcal{U}$ witnessed by $\Phi$ and $\Psi$. Then $X \in \text{MLR}$ and $n \geq \text{rd}_\mathcal{U}(\Phi(X))$ imply $\Psi(n) = \text{rd}_\mathcal{V}(X)$.

Fix $X, Y \in \text{MLR}$ with $\text{rd}_\mathcal{V}(X) \neq \text{rd}_\mathcal{V}(Y)$.

Let $n \geq \max(\text{rd}_\mathcal{U}(\Phi(X)), \text{rd}_\mathcal{U}(\Phi(Y)))$. 
**Theorem.** Let $\mathcal{U}$ and $\mathcal{V}$ be universal ML-tests. Then

$$RD_V \equiv_W LAY_\mathcal{U}.$$  

**Proof.** Same idea as for the previous proof, but more involved.

**Proposition.** Let $\mathcal{U}$ and $\mathcal{V}$ be universal ML-tests. Then

$$RD_V \not\leq_{sW} LAY_\mathcal{U}.$$  

**Proof.** Suppose $RD_V \leq_{sW} LAY_\mathcal{U}$ witnessed by $\Phi$ and $\Psi$. Then $X \in \text{MLR}$ and $n \geq \text{rd}_\mathcal{U}(\Phi(X))$ imply $\Psi(n) = \text{rd}_\mathcal{V}(X)$. Fix $X, Y \in \text{MLR}$ with $\text{rd}_\mathcal{V}(X) \neq \text{rd}_\mathcal{V}(Y)$. Let $n \geq \max(\text{rd}_\mathcal{U}(\Phi(X)), \text{rd}_\mathcal{U}(\Phi(Y)))$. Then $\text{rd}_\mathcal{V}(X) = \Psi(n) = \text{rd}_\mathcal{V}(Y)$.
Conjecture. There exists universal tests $\mathcal{U}$ and $\mathcal{V}$ such that

$$\text{RD}_{\mathcal{U}} \not\equiv_{\text{SW}} \text{RD}_{\mathcal{V}}.$$ 

Article reference.

- Submitted October 2014.
- Available on ArXiv as 1409.8589.
- Contains many other related results.
One last thing
One last thing

Merry Christmas!
One last thing

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One last thing

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