Vitali Covering in the Weihrauch degrees

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Introduction

The Vitali Covering Theorem

1. **Definition.** Let $\mathscr{I} = (I_n)_n$ be a sequence of open intervals $I_n \subseteq \mathbb{R}$, let $x \in \mathbb{R}$ and $A \subseteq \mathbb{R}$.
   - We say that $x \in [0, 1]$ is *captured* by $\mathscr{I}$, if for every $\varepsilon > 0$ there exists some $n \in \mathbb{N}$ with $\operatorname{diam}(I_n) < \varepsilon$ and $x \in I_n$.
   - We say that $\mathscr{I}$ is *saturated*, if every $x \in \bigcup \mathscr{I} := \bigcup_{n=0}^{\infty} I_n$ is captured by $\mathscr{I}$.
   - We call $\mathscr{I}$ a *Vitali cover* of A, if every $x \in A$ is captured by $\mathscr{I}$.
   - We say that $\mathscr{I}$ *eliminates* A, if the I_n are pairwise disjoint and $\lambda(A \setminus \bigcup \mathscr{I}) = 0$.
2. **Theorem (VCT for intervals).** Let $A \subseteq [0, 1]$ be Lebesgue measurable and let $\mathscr{I}$ be a Vitali cover of A. Then there exists a subsequence $\mathscr{J}$ of $\mathscr{I}$ that eliminates A.

VCT in reverse mathematics

1. VCT was studied in reverse mathematics.
2. **Theorem (Brown, Giusto, Simpson).** Over RCA_0, VCT for intervals is equivalent to WWKL_0.
3. Here *Weak Weak König's Lemma* WWKL_0 is the statement that a binary tree of positive measure has a path.
4. Inspired by this work, we study VCT in the Weihrauch degrees.
5. **Motivation.** As seen in recent work by the authors, the Weihrauch degrees may expose more fine structure.

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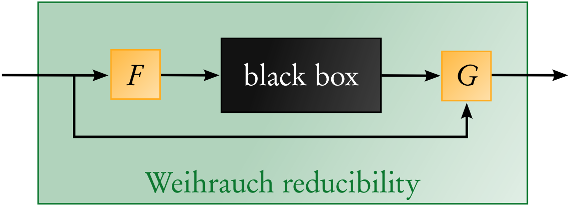
Weihrauch degrees

Weihrauch degrees

1. **General idea.** We look at mathematical tasks where a problem is given to a black box, and the black box has to give back a solution to the problem.
2. **Example.** Given a bounded sequence of rationals, the black box has to produce an accumulation point.
3. **Reducibility.** Assume we have a black box solving a certain problem $\mathscr{A}$. Can we use it to solve another problem $\mathscr{B}$?
4. **Approach.**
    - Code an instance B of problem $\mathscr{B}$ into a valid instance A of problem $\mathscr{A}$.
    - Run the black box for $\mathscr{A}$ on A.
    - Convert the solution for A back into a solution for B.

Weihrauch degrees, formally



1. F and G are both Turing functionals.[1]
2. **Weak reducibility.** The decoding procedure G has access to the original input.
3. That is, if B is the black box, the computed function is

$$x \mapsto G(\langle B(F(x)), x\rangle).$$

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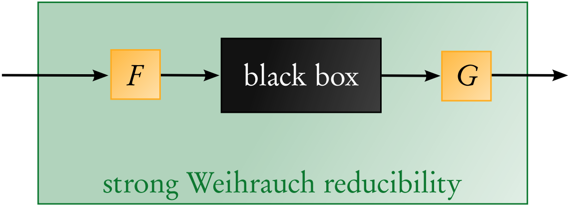
[1] modulo representations.

Weihrauch degrees, formally



1. F and G are both Turing functionals.[1]
2. **Strong reducibility.** The decoding procedure G *does not* have access to the original input.
3. That is, if B is the black box, the computed function is

$$x \mapsto G(B(F(x))).$$

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A fragment of the Weihrauch zoo

1. The reducibilities produce a degree structure of mathematical tasks of different levels of difficulty.

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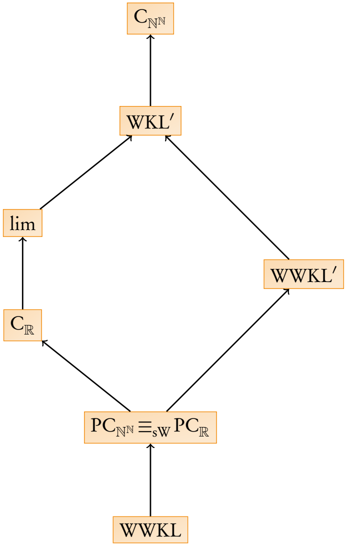

$\mathrm{C}_{\mathbb{N}^\mathbb{N}}$

WKL'

$\lim$

WWKL'

$\mathrm{C}_{\mathbb{R}}$

$\mathrm{PC}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{sW}} \mathrm{PC}_{\mathbb{R}}$

WWKL

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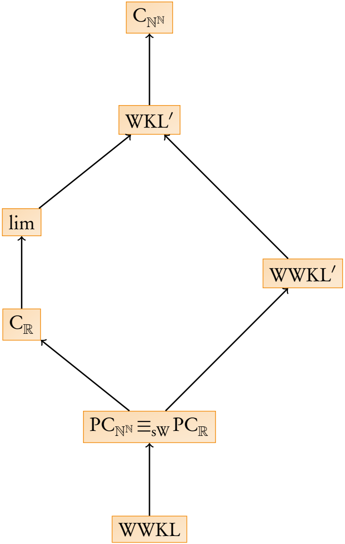

1. The reducibilities produce a degree structure of mathematical tasks of different levels of difficulty.
2. Many tasks are equivalent to *choice principles*.

Choice principles

1. A choice principle is a black box solving a task as follows:
    - A set is described by giving more and more negative information, encoded into an infinite sequence in, say, $\mathbb{N}^{\mathbb{N}}$.
    - That is, step by step we obtain more and more information which elements are *not* in the set.
    - The task is to give an element that will never be removed.
2. **Example.** Choice on the natural numbers $\mathrm{C}_{\mathbb{N}}$.

Choice principles

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Possible black box output: 6

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2. **Example.** Choice on the natural numbers $\mathrm{C}_{\mathbb{N}}$.

   Negative information: 3 0 1 4 9 7 …

   Possible black box output: 6
3. Similarly, we can define $\mathrm{C}_{[0,1]}$, $\mathrm{C}_{2^{\mathbb{N}}}$, $\mathrm{C}_{\mathbb{R}}$, $\mathrm{C}_{\mathbb{N}^{\mathbb{N}}}$,…

Other relevant forms of choice

1. **Positive choice:** $\mathrm{PC}_{[0,1]}$, $\mathrm{PC}_{2^{\mathbb{N}}}$, $\mathrm{PC}_{\mathbb{R}}$, $\mathrm{PC}_{\mathbb{N}^{\mathbb{N}}}, \ldots$
   For every valid instance of the problem there exist positive measure many correct solutions.
2. Brattka and Pauly noticed that $\mathrm{WWKL} = \mathrm{PC}_{2^{\mathbb{N}}} = \mathrm{PC}_{[0,1]}$.
3. **Arithmetic choice:** $\boldsymbol{\Sigma}^0_k\mathrm{C}_{\square}$, $\boldsymbol{\Pi}^0_k\mathrm{C}_{\square}, \ldots$
   Choice for $\boldsymbol{\Sigma}^0_k$ and $\boldsymbol{\Pi}^0_k$ subsets of $\square$, using representations $\delta^{\square}_{\boldsymbol{\Sigma}^0_k}$ and $\delta^{\square}_{\boldsymbol{\Pi}^0_k}$ inductively defined over k (Brattka).
4. **Positive arithmetic choice:** $\mathrm{P}\boldsymbol{\Sigma}^0_k\mathrm{C}_{\square}$, $\mathrm{P}\boldsymbol{\Pi}^0_k\mathrm{C}_{\square}, \ldots$
5. **Limited principle of omniscience:** LPO
   Characteristic function of $0^{\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$.

3

VCT in the Weihrauch degrees

A natural, but trivial version

1. The blackbox model used in the Weihrauch degrees requires that we formulate the Vitali Covering Theorem in such a form.
2. There is more than one way of doing this.
3. **Definition.** By Int we denote the set of sequences $(I_n)_n$ of open Intervals $I_n = (a, b)$ with $a, b \in \mathbb{Q}$.
4. **Definition.** $\mathrm{VCT}_0 :\subseteq \mathrm{Int} \rightrightarrows \mathrm{Int}$ with

   $$\mathrm{VCT}_0(\mathscr{I}) := \{\mathscr{J} : \mathscr{J} \text{ is a subsequence of } \mathscr{I} \text{ that eliminates } [0,1]\}$$

   and $\mathrm{dom}(\mathrm{VCT}_0)$ contains all $\mathscr{I} \in \mathrm{Int}$ that are Vitali covers of $[0, 1]$.

A natural, but trivial version

1. **Theorem (Brattka, Pauly).** VCT_0 is computable.
2. **Proof idea.** Search for a finite sequence of disjoint intervals covering $[0,1]$ up to measure $1/2$. The classical VCT guarantees that we find this.

   Iterate for the *interior* of the remaining uncovered area up to measures $1/4, 1/8, 1/16\ldots$

   This is effective. □
3. So the most straight-forward formulation of VCT is trivial in the Weihrauch degrees.
4. To obtain more interesting results, we need to look at contrapositive versions of VCT.

Two non-trivial versions

1. **Idea.** Given an $\mathscr{I}$ that violates some of the requirements for being a Vitali cover, we want to find an $x \in [0,1]$ witnessing this violation.
2. Again, there is more than one formalisation for this idea, as there are different ways and degrees of violating the requirements.
3. It will turn out that these differences actually effect different computational strengths of the associated mathematical tasks.

Two non-trivial versions

1. **Definition.** $\mathrm{VCT}_1 :\subseteq \mathrm{Int} \rightrightarrows [0,1]$ with

$$\mathrm{VCT}_1(\mathscr{I}) := [0,1] \setminus \bigcup \mathscr{I}$$

   and $\mathrm{dom}(\mathrm{VCT}_1)$ contains all $\mathscr{I} \in \mathrm{Int}$ that are saturated and that do not have a subsequence that eliminates $[0,1]$.

2. **Definition.** $\mathrm{VCT}_2 :\subseteq \mathrm{Int} \rightrightarrows [0,1]$ with

$$\mathrm{VCT}_2(\mathscr{I}) := \{x \in [0,1] : x \text{ is not captured by } \mathscr{I}\}$$

   and $\mathrm{dom}(\mathrm{VCT}_2)$ contains all $\mathscr{I} \in \mathrm{Int}$ that do not have a subsequence that eliminates $[0,1]$.

Characterisation of VCT_1

1. **Lemma (Vitalisation).** There exists a computable map $V : \mathrm{Int} \to \mathrm{Int}$ such that $\bigcup \mathscr{I} = \bigcup V(\mathscr{I})$ for all $\mathscr{I} \in \mathrm{Int}$ and $\mathrm{range}(V)$ only consists of saturated sequences of intervals. □
2. **Theorem.** $\mathrm{VCT}_1 \equiv_{\mathrm{sW}} \mathrm{WWKL}$.

   Proof idea. Show equivalence with $\mathrm{PC}_{[0,1]}$.

   To see $\mathrm{PC}_{[0,1]} \leq_{\mathrm{sW}} \mathrm{VCT}_1$, take an instance for $\mathrm{PC}_{[0,1]}$ and apply vitalisation to obtain an instance of VCT_1.

   Inversely, let $\mathscr{I}$ be a saturated sequence of intervals not having a subsequence that eliminates $[0, 1]$.
   Compute $A := [0, 1] \setminus \bigcup \mathscr{I}$.
   As $\mathscr{I}$ is a Vitali cover of $\bigcup \mathscr{I}$, by the (classical) VCT there is a $\mathscr{J} \subseteq \mathscr{I}$ eliminating $\bigcup \mathscr{I}$.
   If $\mu(A) = 0$, then $\mathscr{J}$ also eliminates $[0, 1]$, contradiction.
   Hence $\mu(A) > 0$ and $\mathrm{VCT}_1(\mathscr{I}) = \mathrm{PC}_{[0,1]}(A)$. □

Locating VCT_2

1. **Proposition.** $\mathrm{VCT}_1 \leq_{\mathrm{sW}} \mathrm{VCT}_2$.

   Proof. If $\mathscr{I}$ is a saturated sequence of rational open intervals containing no subsequence eliminating $[0,1]$, then $\mathscr{I}$ does not cover $[0,1]$ and every point $x \in [0,1]$ not captured by $\mathscr{I}$ is not covered by $\mathscr{I}$, i.e. $x \in [0,1] \setminus \bigcup \mathscr{I}$.
   Hence VCT_1 is a restriction of VCT_2. $\square$

2. **Proposition.** $\mathrm{LPO} \leq_{\mathrm{sW}} \mathrm{VCT}_2$

   Proof idea. Initially start vitalising $\left[0, \frac{1}{4}\right]$.
   If reading a non-zero symbol, stop this.
   Instead start vitalising $\left[\frac{3}{4}, 1\right]$. $\square$

3. **Theorem.** $\mathrm{VCT}_0 <_{\mathrm{W}} \mathrm{VCT}_1 <_{\mathrm{W}} \mathrm{VCT}_2$.

   Proof idea. Use $\mathrm{LPO} \not\leq_{\mathrm{W}} \mathrm{WKL}$ (Brattka, Gherardi). $\square$

4. **Theorem.** $\mathrm{VCT}_2 \mid_{\mathrm{W}} \mathrm{WKL}$

Locating VCT_2

1. **Theorem.** $\mathrm{VCT}_2 \equiv_{\mathrm{sW}} \mathrm{PC}_{\mathbb{R}}$.

   Proof idea. Note that

$$\mathrm{PC}_{\mathbb{R}} \equiv_{\mathrm{sW}} \mathrm{C}_{\mathbb{N}} \times \mathrm{WWKL} \equiv_{\mathrm{sW}} \mathrm{C}_{\mathbb{N}} \times \mathrm{VCT}_1.$$

   So it suffices to show

   - $\mathrm{C}_{\mathbb{N}} \times \mathrm{VCT}_1 \leq_{\mathrm{sW}} \mathrm{VCT}_2$, and
   - $\mathrm{VCT}_2 \leq_{\mathrm{sW}} \mathrm{PC}_{\mathbb{R}}$.

Locating VCT_2

1. **Proof idea (continued).** First we show $\mathrm{C}_{\mathbb{N}} \times \mathrm{VCT}_1 \leq_{\mathrm{sW}} \mathrm{VCT}_2$, using an idea of Pauly.

(→ blackboard)

Almost Vitali Covers

1. **Proof idea (continued).** To see that $\mathrm{VCT}_2 \leq_{\mathrm{sW}} \mathrm{PC}_{\mathbb{R}}$, look at *almost Vitali covers*.
2. **Definition (Almost Vitali cover).** A sequence of intervals $\mathscr{I}$ is an *almost Vitali cover* of a Lebesgue measurable set $A \subseteq [0,1]$ if for all $\varepsilon > 0$ and

$$U_\varepsilon := \bigcup \{I : I \in \mathscr{I};\ \mathrm{diam}(I_n) < \varepsilon\}.$$

   it holds that $\lambda(A \setminus U_\varepsilon) = 0$.
3. **Intuition.** If $\mathscr{I}$ is *not* an almost Vitali cover, then this becomes apparent by excluding long intervals.

Almost Vitali Covers

1. **Definition.** $\mathrm{NAVC}_{[0,1]} :\subseteq \mathrm{Int} \rightrightarrows [0,1]$ with

   $$\mathrm{NAVC}_{[0,1]}(\mathscr{I}) := \{x \in [0,1] : x \text{ is not captured by } \mathscr{I}\}$$

   and $\mathrm{dom}(\mathrm{NAVC}_{[0,1]})$ contains all $\mathscr{I} \in \mathrm{Int}$ that are not almost Vitali covers of $[0,1]$.
2. A theorem of Brown, Giusto, and Simpson implies that instances of VCT_2 are in particular instances of $\mathrm{NAVC}_{[0,1]}$.
3. So it remains to show that $\mathrm{NAVC}_{[0,1]} \leq_{\mathrm{sW}} \mathrm{PC}_{\mathbb{R}}$.

Almost Vitali Covers

1. **Theorem.** $\mathrm{NAVC}_{[0,1]} \equiv_{\mathrm{sW}} \mathrm{P}\Sigma^0_2\mathrm{C}_{[0,1]}$

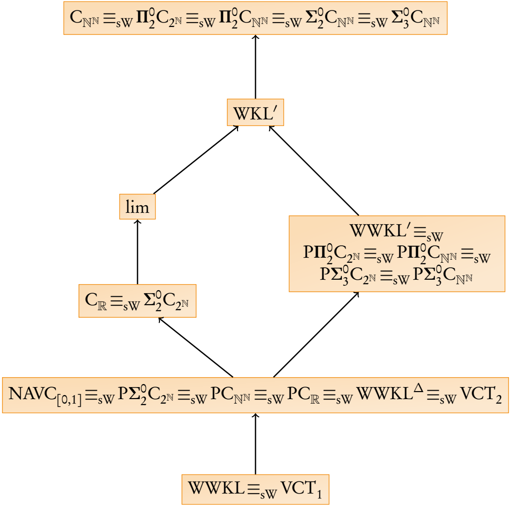
   Proof idea. $\mathrm{NAVC}_{[0,1]} \leq_{\mathrm{sW}} \mathrm{P}\Sigma^0_2\mathrm{C}_{[0,1]}$ is clear.

   Inversely, approximate the complement via $\bigcap_{i \leq s} O_i[t]$.
   Start vitalising areas that currently look covered.
   If they later become uncovered again, suspend vitalisation, etc.
   Care is needed with regards to timing and interval lengths used in vitalisation. ☐
2. The reverse maths analogue of $\mathrm{NAVC}_{[0,1]}$ is provable in WWKL_0 by Brown, Giusto, Simpson.
3. So we wondered how $\mathrm{P}\Sigma^0_2\mathrm{C}_{[0,1]}$ compares to $\mathrm{PC}_{\mathbb{R}}$, which led us to study…

4

Arithmetic choice principles

Enriched zoo



$C_{\mathbb{N}^\mathbb{N}} \equiv_{sW} \Pi^0_2 C_{2^\mathbb{N}} \equiv_{sW} \Pi^0_2 C_{\mathbb{N}^\mathbb{N}} \equiv_{sW} \Sigma^0_2 C_{\mathbb{N}^\mathbb{N}} \equiv_{sW} \Sigma^0_3 C_{\mathbb{N}^\mathbb{N}}$

WKL'

$\lim$

$\mathrm{WWKL}' \equiv_{sW} \mathrm{P}\Pi^0_2 C_{2^\mathbb{N}} \equiv_{sW} \mathrm{P}\Pi^0_2 C_{\mathbb{N}^\mathbb{N}} \equiv_{sW} \mathrm{P}\Sigma^0_3 C_{2^\mathbb{N}} \equiv_{sW} \mathrm{P}\Sigma^0_3 C_{\mathbb{N}^\mathbb{N}}$

$C_\mathbb{R} \equiv_{sW} \Sigma^0_2 C_{2^\mathbb{N}}$

$\mathrm{NAVC}_{[0,1]} \equiv_{sW} \mathrm{P}\Sigma^0_2 C_{2^\mathbb{N}} \equiv_{sW} \mathrm{P}C_{\mathbb{N}^\mathbb{N}} \equiv_{sW} \mathrm{P}C_\mathbb{R} \equiv_{sW} \mathrm{WWKL}^\Delta \equiv_{sW} \mathrm{VCT}_2$

$\mathrm{WWKL} \equiv_{sW} \mathrm{VCT}_1$

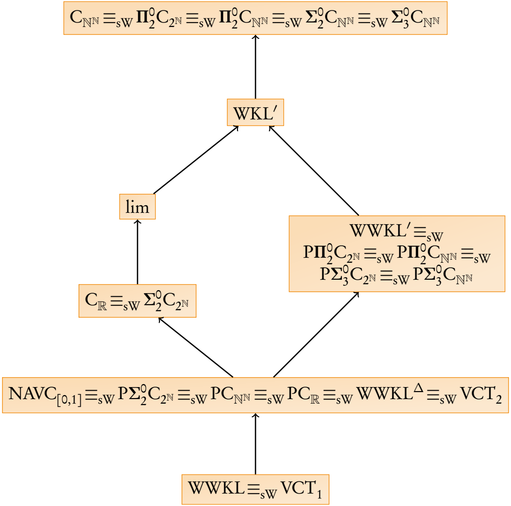
1. In particular, we showed

$$\mathrm{P}\Sigma^0_2 C_{2^\mathbb{N}} \equiv_{sW} \mathrm{P}C_\mathbb{R},$$

completing the proof that $\mathrm{VCT}_2 \equiv_{sW} \mathrm{P}C_\mathbb{R}$. ☐

2. **Open question.**
Is $C_{\mathbb{N}^\mathbb{N}} \equiv_{sW} \boldsymbol{\Pi}^0_3 C_{\mathbb{N}^\mathbb{N}}$?

Enriched zoo



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$\mathrm{WWKL}' \equiv_{sW} \mathrm{P}\boldsymbol{\Pi}^0_2 C_{2^\mathbb{N}} \equiv_{sW} \mathrm{P}\boldsymbol{\Pi}^0_2 C_{\mathbb{N}^\mathbb{N}} \equiv_{sW} \mathrm{P}\boldsymbol{\Sigma}^0_3 C_{2^\mathbb{N}} \equiv_{sW} \mathrm{P}\boldsymbol{\Sigma}^0_3 C_{\mathbb{N}^\mathbb{N}}$

$C_\mathbb{R} \equiv_{sW} \boldsymbol{\Sigma}^0_2 C_{2^\mathbb{N}}$

$\mathrm{NAVC}_{[0,1]} \equiv_{sW} \mathrm{P}\boldsymbol{\Sigma}^0_2 C_{2^\mathbb{N}} \equiv_{sW} \mathrm{PC}_{\mathbb{N}^\mathbb{N}} \equiv_{sW} \mathrm{PC}_\mathbb{R} \equiv_{sW} \mathrm{WWKL}^\Delta \equiv_{sW} \mathrm{VCT}_2$

$\mathrm{WWKL} \equiv_{sW} \mathrm{VCT}_1$

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   completing the proof that $\mathrm{VCT}_2 \equiv_{sW} \mathrm{PC}_\mathbb{R}$. □
2. **Open question.** Is $C_{\mathbb{N}^\mathbb{N}} \equiv_{sW} \boldsymbol{\Pi}^0_3 C_{\mathbb{N}^\mathbb{N}}$?

Thank you for your attention.