Time-bounded Kolmogorov complexity and Solovay functions

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Abstract. A Solovay function is a computable upper bound g for prefix-free Kolmogorov complexity K that is nontrivial in the sense that g agrees with K, up to some additive constant, on infinitely many places n. We obtain natural examples of Solovay functions by showing that for some constant c_0 and all computable functions t such that $c_0 n \leq t(n)$, the time-bounded version K^t of K is a Solovay function.
By unifying results of Bienvenu and Downey and of Miller, we show that a right-computable upper bound g of K is a Solovay function if and only if Ω_g is Martin-Löf random. Letting $\Omega_g = \sum 2^{-g(n)}$, we obtain as a corollary that the Martin-Löf randomness of the various variants of Chaitin's Ω extends to the time-bounded case in so far as Ω_{K^t} is Martin-Löf random for any t as above.
As a step in the direction of a characterization of K-triviality in terms of jump-traceability, we demonstrate that a set A is K-trivial if and only if A is $\mathrm{O}(g(n) - \mathrm{K}(n))$-jump traceable for all Solovay functions g, where the equivalence remains true when we restrict attention to functions g of the form K^t, either for a single or all functions t as above.
Finally, we investigate the plain Kolmogorov complexity C and its time-bounded variant C^t of initial segments of computably enumerable sets. Our main theorem here is a dichotomy similar to Kummer's gap theorem and asserts that every high c.e. Turing degree contains a c.e. set B such that for any computable function t there is a constant $c_t > 0$ such that for all m it holds that $\mathrm{C}^t(B \upharpoonright m) \geq c_t \cdot m$, whereas for any nonhigh c.e. set A there is a computable time bound t and a constant c such that for infinitely many m it holds that $\mathrm{C}^t(A \upharpoonright m) \leq \log m + c$. By similar methods it can be shown that any high degree contains a set B such that $\mathrm{C}^t(B \upharpoonright m) \geq^+ m/4$. The constructed sets B have low unbounded but high time-bounded Kolmogorov complexity, and accordingly we obtain an alternative proof of the result due to Juedes, Lathrop, and Lutz [JLL] that every high degree contains a strongly deep set.

1 Introduction and overview

Prefix-free Kolmogorov complexity K it not computable and in fact does not even allow for computable lower bounds. However, there are computable upper bounds for K and, by a construction that goes back to Solovay [BD,S], there are even computable upper bounds that are nontrivial in the sense that g agrees with K, up to some additive constant, on infinitely many places n; such upper bounds are called Solovay functions.

For any computable time-bound t, the time-bounded version K^t of K is obviously a computable upper bound for K, and we show that K^t is indeed a Solovay function in case $c_0 n \leq t(n)$ for some appropriate constant c_0. As a corollary, we obtain that the Martin-Löf randomness of the various variants of Chaitin's Ω extends to the time-bounded case in so far as for any t as above, the real number

$$\Omega_{\mathrm{K}^t} = \sum_{n \in \mathbb{N}} \frac{1}{2^{\mathrm{K}^t(n)}}$$

is Martin-Löf random. The corresponding proof exploits the result by Bienvenu and Downey [BD] that a computable function g such that $\Omega_g = \sum 2^{-g(n)}$ converges is a Solovay function if and only if Ω_g is Martin-Löf random. In fact, this equivalence extends by an even simpler proof to the case of functions g that are just right-computable, i.e., effectively approximable from above, and one then obtains as special cases the result of Bienvenu and Downey and a related result of Miller where the role of g is played by the fixed right-computable but noncomputable function K.

An open problem that received some attention recently [BDG,DH,N] is whether the class of K-trivial sets coincides with the class of sets that are $g(n)$-jump-traceable for all computable functions g such that $\sum 2^{-g(n)}$ converges. As a step in the direction of a characterization of K-triviality in terms of jump-traceability, we demonstrate that a set A is K-trivial if and only if A is $\mathrm{O}(g(n) - \mathrm{K}(n))$-jump traceable for all Solovay functions g, where the equivalence remains true when we restrict attention to functions g of the form K^t, either for a single or all functions t as above.

Finally, we consider the time-bounded and unbounded Kolmogorov complexity of the initial segments of sets that are computationally enumerable, or c.e., for short. The initial segments of a c.e. set A have small Kolmogorov complexity, more precisely, by Barzdins' lemma it holds that $\mathrm{C}(A \restriction m) \leq^+ 2 \log m$, where C denotes plain Kolmogorov complexity. Theorem 16, our main result in this section, has a structure similar to Kummer's gap theorem in so far as it asserts a dichotomy in the complexity of initial segments between high and nonhigh c.e. sets. More precisely, every high c.e. Turing degree contains a c.e. set B such that for any computable function t there is a constant $c_t > 0$ such that for all m it holds that $\mathrm{C}^t(B \restriction m) \geq c_t \cdot m$, whereas for any nonhigh c.e. set A there is a computable time bound t and a constant c such that for infinitely many m it holds that $\mathrm{C}^t(A \restriction m) \leq \log m + c$. By similar methods it can be shown that any high degree contains a set B such that $\mathrm{C}^t(B \restriction m) \geq^+ m/4$. The constructed sets B have low unbounded but high time-bounded Kolmogorov complexity, and accordinlgy we obtain an alternative proof of the result due to Juedes, Lathrop, and Lutz [JLL] that every high degree contains a strongly deep set.

Notation In order to define plain and prefix-free Kolmogorov complexity, we fix additively optimal oracle Turing machines $\mathbb{V}$ and $\mathbb{U}$, where $\mathbb{U}$ has prefix-free domain. We let $\mathrm{C}^A(x)$ denote the Kolmogorov-complexity of x with respect to $\mathbb{V}$ relative to oracle A, let $\mathrm{C}(x) = \mathrm{C}^{\emptyset}(x)$, and similarly define prefix-free Kolmogorov complexity K^A and K with respect to $\mathbb{U}$. In connection with the definition of time-bounded Kolmogorov complexity, we assume that $\mathbb{V}$ and $\mathbb{U}$ both are able to simulate any other Turing machine M running for t steps in $\mathrm{O}(t \cdot \log(t))$ steps for an arbitrary machine M and in $\mathrm{O}(t(n))$ steps in case M has only two work tapes.

For a computable function $t : \mathbb{N} \to \mathbb{N}$ and a machine M, the Kolmogorov complexity relative to M with time bound t is

$$\mathrm{C}^t_M(n) := \min\{|\sigma| : M(\sigma) \downarrow= n \text{ in at most } t(|n|) \text{ steps}\},$$

and we write C^t for $\mathrm{C}^t_{\mathbb{U}}$. The prefix-free Kolmogorov complexity with time bound t denoted by $\mathrm{K}^t_M(n)$ and $\mathrm{K}^t(n) = \mathrm{K}^t_{\mathbb{U}}$ is defined likewise by considering only prefix-free machines and the corresponding universal machine $\mathbb{U}$ in place of $\mathbb{U}$.

We identify strings with natural numbers by the order isomorphism between the length-lexicographical order on strings and the usual order on $\mathbb{N}$, and we write $|m|$ for the length of the string that corresponds to the number m, where then $|m|$ is roughly $\log m$.

2 Solovay functions and Martin-Löf randomness

Definition 1 (Li, Vitányi [LV]). *A computable function $f : \mathbb{N} \to \mathbb{N}$ is called a* Solovay function *if $\mathrm{K}(n) \leq^+ f(n)$ for all n and $\mathrm{K}(n) =^+ f(n)$ for infinitely many n.*

Solovay [S,BD] had already constructed Solovay functions and by slightly varying the standard construction, next we observe that time-bounded prefix-free Kolmogorov complexity indeed provides natural examples of Solovay functions.

Theorem 2. *There is a constant c_0 such that time-bounded prefix-free Kolmogorov complexity K^t is a Solovay function for any computable function $t \colon \mathbb{N} \to \mathbb{N}$ such that $c_0 n \leq t(n)$ holds for almost all n.*

Proof. Fix a standard effective and effectively invertible pairing function $\langle .,. \rangle \colon \mathbb{N}^2 \to \mathbb{N}$ and define a tripling function $[.,.,.] \colon \mathbb{N}^3 \to \mathbb{N}$ by letting

$$[s, \sigma, n] = 1^s 0 \langle \sigma, n \rangle.$$

Let M be a Turing machine with two tapes that on input σ uses its first tape to simulate the universal machine $\mathbb{U}$ on input σ and, in case $\mathbb{U}(\sigma) = n$, to compute $\langle \sigma, n \rangle$, while maintaining on the second tape a unary counter for the number of steps of M required for these computations. In case eventually $\langle \sigma, n \rangle$

had been computed with final counter value s, the output of M is $z = [s, \sigma, n]$, where by construction in this case the total running time of M is in $\mathrm{O}(s)$.

Call z of the form $[s, \sigma, n]$ a Solovay triple in case $M(\sigma) = z$ and σ is an optimal code for n, i.e., $\mathrm{K}(n) = |\sigma|$. For some appropriate constant c_0 and any computable function t that eventually is at least $c_0 n$, for almost all such triples z it then holds that

$$\mathrm{K}(z) =^{+} \mathrm{K}^{t}(z),$$

because given a code for M and σ, by assumption the universal machine $\mathbb{U}$ can simulate the computation of the two-tape machine M with input σ with linear overhead, hence $\mathbb{U}$ uses time $\mathrm{O}(s)$ plus the constant time required for decoding M, i.e., time at most $c_0|z|$. □

Next we derive a unified form of a characterization of Solovay function in terms of Martin-Löf randomness of the corresponding Ω-number due to Bienvenu and Downey [BD] and a result of Miller [M] that asserts that the notions of weakly low and low for Ω coincide. Before, we review some standard notation and facts relating to Ω-numbers.

Definition 3. *For a function $f\colon \mathbb{N} \to \mathbb{N}$, the* Ω-number *of f is*

$$\Omega_f := \sum_{n \in \mathbb{N}} 2^{-f(n)}$$

Definition 4. *A function $f\colon \mathbb{N} \to \mathbb{N}$ is an* information content measure relative to a set *A in case f is right-computable with access to the oracle A and Ω_f converges; furthermore, the function f is an* information content measure *if it is an information content measure relative to the empty set.*

The following remark describes for a given information content measure f an approximation from below to Ω_f that has certain special properties. For the sake of simplicity, in the remark only the oracle-free case is considered and the virtually identical considerations for the general case are omitted.

Remark 5. For a given information content measure f, we fix as follows a non-decreasing computable sequence $a_0, a_1, \ldots$ that converges to Ω_f and call this sequence the canonical approximation of Ω_f.

First, we fix some standard approximation to the given information content measure f from above, i.e., a computable function $(n, s) \mapsto f_s(n)$ such that for all n the sequence $f_0(n), f_1(n), \ldots$ is a nonascending sequence of natural numbers that converges to $f(n)$, where we assume in addition that $f_s(n) - f_{s+1}(n) \in \{0, 1\}$. Then in order to obtain the a_i, let $a_0 = 0$ and given a_i, define a_{i+1} by searching for the next pair of the form $(n, 0)$ or the form $(n, s+1)$ where in addition it holds that $f_s(n) - f_{s+1}(n) = 1$ (with some ordering of pairs understood), let

$$d_i = 2^{-f_0(n)} \qquad \text{or} \qquad d_i = 2^{-f_{s+1}(n)} - 2^{-f_s(n)} = 2^{-f_s(n)} \,,$$

respectively, and let $a_{i+1} = a_i + d_i$. Furthermore, in this situation, say that the increase of d_i from a_i to a_{i+1} occurs due to n.

It is well-known [DH] that among all right-computable functions exactly the information content measures are, up to an additive constant, upper bounds for the prefix-free Kolmogorov complexity K.

Theorem 6 unifies two results by Bienvenu and Downey [BD] and by Miller [M], which are stated below as Corollaries 7 and 8. The proof of the backward direction of the equivalence stated in Theorem 6 is somewhat more direct and uses different methods when compared to the proof of Bienvenu and Downey, and is quite a bit shorter than Miller's proof, though the main trick of delaying the enumeration via the notion of a matched increase is already implicit there [DH,M]. Note in this connection that Bienvenu has independently shown that Miller's result can be obtained as a corollary to the result of Bienvenu and Downey [DH].

Theorem 6. *Let f be an information content measure relative to a set A. Then f has the Solovay property with respect to K^A, i.e.,*

$$\lim_{n\to\infty}(f(n)-\mathrm{K}^A(n)) \neq +\infty \tag{1}$$

if and only if Ω_f is Martin-Löf random relative to A.

Proof. We first show the backwards direction of the equivalence asserted in the theorem, where the construction and its verification bear some similarities to Kučera and Slaman's [KS] proof that left-computable sets that are not Solovay complete cannot be Martin-Löf random. We assume that (1) is false and construct a sequence $U_0, U_1, \ldots$ of sets that is a Martin-Löf test relative to A and covers Ω_f. In order to obtain the component U_c, let $a_0, a_1, \ldots$ be the canonical approximation to Ω_f where in particular $a_{i+1} = a_i + d_i$ for increases d_i that occur due to some n. Let b_i be the sum of all increases d_j such that $j \leq i$ and where d_j and d_i are due to the same n. With the index c understood, say an increase d_i due to n is matched if it holds that

$$2^{c+1} b_i \leq 2^{-\mathrm{K}^A(n)}.$$

For every d_i for which it could be verified that d_i is matched, add an interval of size $2d_i$ to U_c where this interval either starts at a_i or at the maximum place that is already covered by U_c, whichever is larger. By construction the sum of all matched d_i is at most $\Omega^A/2^{c+1} \leq 2^{-(c+1)}$ and the sets U_c are uniformly c.e. relative to A, hence $U_0, U_1, \ldots$ is a Martin-Löf test relative to A. Furthermore, this test covers Ω_f because by the assumption that (1) is false, for any c almost all increases are matched.

For ease of reference, we review the proof of the forward direction of the equivalence asserted in the theorem, which follows by the same line of standard argument that has already been used by Bienvenu and Downey and by Miller. For a proof by contraposition, assume that Ω_f is not Martin-Löf random relative to A, i.e., for every constant c there is a prefix σ_c of Ω_f such that $\mathrm{K}^A(\sigma_c) \leq |\sigma_c| - 2c$. Again consider the canonical approximation $a_0, a_1, \ldots$ to Ω_f where $f(n,0), f(n,1), \ldots$ is the corresponding effective approximation from above to $f(n)$ as in Remark 5. Moreover, for σ_c as above we let s_c be the least index s such that a_s exceeds σ_c

(assuming that the expansion of Ω_f is not eventually constant and leaving the similar considerations for this case to the reader). Then the sum over all values $2^{-f(n)}$ such that none of the increases d_0 through d_s was due to n is at most $2^{-|\sigma_c|}$, hence all pairs of the form $(f(n,s) - |\sigma_c| + 1, n)$ for such n and s where either $s = 0$ or $f(n,s)$ differs from $f(n, s-1)$ form a sequence of Kraft-Chaitin axioms, which is uniformly effective in c and σ_c relative to oracle A. Observe that by construction for each n, there is an axiom of the form $(f(n) - |\sigma_c| + 1, n)$ and the sum of all terms 2^{-k} over all axioms of the form (k, n) is less than $2^{-f(n)-|\sigma_c|}$.

Now consider a prefix-free Turing machine M with oracle A that given codes for c and σ_c and some other word p as input, first computes c and σ_c, then searches for s_c, and finally outputs the word that is coded by p according to the Kraft-Chaitin axioms for c, if such a word exists. If we let d be the coding constant for M, we have for all sufficiently large c and x that $\mathrm{K}^A(n) \leq 2 \log c + \mathrm{K}^A(\sigma_c) + f(n) - |\sigma_c| + 1 + d \leq f(n) - c$. □

As special cases of Theorem 6 we obtain the following results by Bienvenu and Downey [BD] and by Miller [M], where the former one is immediate and for the latter one it suffices to observe that the definition of the notion low for Ω in terms of Chaitin's Ω number

$$\Omega := \sum_{\{x\colon\, \mathbb{U}(x)\downarrow\}} 2^{-|x|}.$$

is equivalent to a definition in terms of Ω_{K}.

Corollary 7 (Bienvenu and Downey). *A computable information content measure f is a Solovay function if and only if Ω_f is Martin-Löf random.*

Corollary 8 (Miller). *A set A is weakly low if and only if A is low for Ω.*

Proof. In order to see the latter result, it suffices to let $f = \mathrm{K}$ and to recall that for this choice of f the properties of A that occur in the two equivalent assertions in the conclusion of Theorem 6 coincide with the concepts weakly low and low for Ω_K. But the latter property is equivalent to being low for Ω, since for any set A, it is equivalent to require that some or that all left-computable Martin-Löf random set are Martin-Löf random relative to A [N, Proposition 8.8.1]. □

By Corollary 7 and Theorem 2 it is immediate that the known Martin-Löf randomness of Ω_{K} extends to the time-bounded case.

Corollary 9. *There is a constant c_0 such that $\Omega_{\mathrm{K}^t} := \sum_{x \in \mathbb{N}} 2^{-\mathrm{K}^t(x)}$ is Martin-Löf random for any computable function t where $c_0 n \leq t(n)$ for almost all n.*

3 Solovay functions and jump-traceability

In an attempt to define K-triviality without resorting to effective randomness or measure, Barmpalias, Downey and Greenberg [BDG] searched for characterizations of K-triviality via jump-traceability. They demonstrated that K-triviality is not implied by being h-jump-traceable for all computable functions h such that $\sum_n 1/h(n)$ converges. Subsequently, the following question received some attention: Can K-triviality be characterized by being g-jump traceable for all computable functions g such that $\sum 2^{-g(n)}$ converges, that is, for all computable functions g that, up to an additive constant term, are upper bounds for K?

We will now argue that Solovay functions can be used for a characterization of K-triviality in terms of jump traceability. However, we will not be able to completely avoid the notion of Kolmogorov complexity.

Definition 10. *A set A is* K-trivial *if* $\mathrm{K}(A \upharpoonright n) \leq^+ \mathrm{K}(n)$ *for all n.*

Definition 11. *Let $h : \mathbb{N} \to \mathbb{N}$ be a computable function. A set A is* $\mathrm{O}(h(n))$-jump-traceable *if for every function Φ partially computable in A there is a function $h \in \mathrm{O}(h(n))$ and a sequence $(T_n)_{n \in \mathbb{N}}$ of uniformly c.e. finite sets, which is called a trace, such that for all n*

$$|T_n| \leq h(n), \quad \text{and} \quad \Phi(n) \in T_n$$

for all n such that $\Phi(n)$ is defined.

Theorem 12. *There is a constant c_0 such that the following assertions are equivalent for any set A.*

(i) *A is* K*-trivial.*
(ii) *A is $\mathrm{O}(g(n) - \mathrm{K}(n))$-jump-traceable for every Solovay function g.*
(iii) *A is $\mathrm{O}(\mathrm{K}^t(n) - \mathrm{K}(n))$-jump-traceable for all computable functions t where $c_0 n \leq t(n)$ for almost all n.*
(iv) *A is $\mathrm{O}(\mathrm{K}^t(n) - \mathrm{K}(n))$-jump-traceable for some computable function t where $c_0 n \leq t(n)$ for almost all n.*

Proof. The implication (ii)⇒(iii) is immediate by Theorem 2, and the implication (iii)⇒(iv) is trivially true. So it suffices to show the implications (i)⇒(ii) and (iv)⇒(i), where due to space considerations we only sketch the corresponding proofs.

First, let A be K-trivial and let Φ^A be any partially A-computable function. Let $\langle .,. \rangle$ be some standard effective pairing function. Since A is K-trivial and hence low for K, we have

$$\mathrm{K}(\langle n, \Phi^A(n) \rangle =^+ \mathrm{K}^A(\langle n, \Phi^A(n) \rangle =^+ \mathrm{K}^A(n) =^+ \mathrm{K}(n)\ ,$$

whenever $\Phi^A(n)$ is defined. Observe that the constant that is implicit in the relation $=^+$ depends only on A in the case of the first and last relation symbol, but depends also on Φ in case of the middle one.

By the coding theorem there can be at most constantly many pairs of the form (n, y) such that $\mathrm{K}(n, y)$ and $\mathrm{K}(n)$ differ at most by a constant, and given n, $\mathrm{K}(n)$ and the constant, we can enumerate all such pairs. But then given a Solovay function g, for given n we can enumerate at most $g(n) - \mathrm{K}(n) + 1$ possible values for $\mathrm{K}(n)$ and for each such value at most constantly many pairs (n, y) such that some y is equal to $\Phi^A(n)$, hence A is $\mathrm{O}(g(n) - \mathrm{K}(n))$-jump-traceable.

Next, let c_0 be the constant from Theorem 2 and let t be a computable time bound such that (iv) is true for this value of c_0. Then K^t is a Solovay function by choice of c_0.

Recall the tripling function $[.,.,.]$ and the concept of a Solovay triple $[s, \sigma, n]$ from the proof of Theorem 2, and define a partial A-computable function Φ that maps any Solovay triple $[s, \sigma, n]$ to $A \upharpoonright n$. Then given an optimal code σ for n, one can compute the corresponding Solovay triple $z = [s, \sigma, n]$, where then $\mathrm{K}^t(z)$ and $\mathrm{K}(z)$ differ only by a constant, hence the trace of Φ^A at z has constant size and contains the value $A \upharpoonright n$, i.e., we have $\mathrm{K}(A \upharpoonright n) \leq^+ |\sigma| = \mathrm{K}(n)$, hence A is K-trivial. □

4 Time bounded Kolmogorov complexity and strong depth

The initial segments of a c.e. set A have small Kolmogorov complexity, by Barzdins' lemma [DH] it holds for all m that

$$\mathrm{C}(A \upharpoonright m \mid m) \leq^+ \log m \quad \text{and} \quad \mathrm{C}(A \upharpoonright m) \leq^+ 2 \log m.$$

Furthermore, there are infinitely many initial segments that have considerably smaller complexity. The corresponding observation in the following remark is extremely easy, but apparently went unnoticed so far and, in particular, improves on corresponding statements in the literature [DH, Lemma attributed to Solovay in Chapter 14].

Remark 13. Let A be a c.e. set. Then there is a constant c such that for infinitely many m it holds that

$$\mathrm{C}(A \upharpoonright m \mid m) \leq c, \quad \mathrm{C}(A \upharpoonright m) \leq^+ \mathrm{C}(m) + c, \quad \text{and } \mathrm{C}(A \upharpoonright m) \leq \log m + c.$$

For a proof, it suffices to fix an effective enumeration of A and to observe that there are infinitely many $m \in A$ such that m is enumerated after all numbers $n \leq m$ that are in A, i.e., when knowing m one can simulate the enumeration until m appears, at which point one then knows $A \upharpoonright m$.

Barzdins [Ba] states that there are c.e. sets with high time-bounded Kolmogorov complexity, and the following lemma generalizes this in so far as such sets can be found in every high Turing degree.

Lemma 14. *For any high set A there is a set B where $A =_{\mathrm{T}} B$ such that for every computable time bound t there is a constant $c_t > 0$ where*

$$\mathrm{C}^t(B \upharpoonright m) \geq^+ c_t \cdot m \quad \text{and} \quad \mathrm{C}(B \upharpoonright m) \leq^+ 2 \log m.$$

Moreover, if A is c.e., B can be chosen to be c.e. as well.

Proof. Let A be any high set. We will construct a Turing-equivalent set B as required. Recall the following equivalent characterization of a set A being high: there is a function g computable in A that majorizes any computable function f, i.e., $f(n) \leq g(n)$ for almost all n. Fix such a function g, and observe that in case A is c.e., we can assume that g can be effectively approximated from below. Otherwise we may replace g with the function g' defined as follows. Let M_g be an oracle Turing machine that computes g if supplied with oracle A. For all n, let

$$\tilde{g}(n, s) := \max\{M_g^{A_i}(n) \mid i \leq s\},$$

where A_i is the approximation to A after i steps of enumeration, and let $g'(n) := \lim_{s\to\infty} \tilde{g}(n, s)$. We have $g(n) \leq g'(n)$ for all n and by construction, g' can be effectively approximated from below.

Partition $\mathbb{N}$ into consecutive intervals $I_0, I_1, \ldots$ where interval I_j has length 2^j and let $m_j = \max I_j$. By abuse of notation, let $t_0, t_1, \ldots$ be an effective enumeration of all partial computable functions. Observe that it is sufficient to ensure that the assertion in the theorem is true for all $t = t_i$ such that t_i is computable, nondecreasing and unbounded. Assign the time bounds to the intervals $I_0, I_1, \ldots$ such that t_0 will be assigned to every second interval including the first one, t_1 to every second interval including the first one of the *remaining* intervals, and so on for $t_2, t_3, \ldots$, and note that this way t_i will be assigned to every 2^{i+1}-th interval.

We construct a set B as required. In order to code A into B, for all j let $B(m_j) = A(j)$, while the remaining bits of B are specified as follows. Fix any interval I_j and assume that this interval is assigned to $t = t_i$. Let B have empty intersection with $I_j \setminus \{m_j\}$ in case the computation of $t(m_j)$ requires more than $g(j)$ steps. Otherwise, run all codes of length at most $|I_j| - 2$ on the universal machine $\mathbb{V}$ for $2t(m_j)$ steps each, and let w_j be the least word of lenght $|I_j| - 1$ that is not output by any of these computations, hence $\mathrm{C}^{2t}(w_j) \geq |w_j| - 1$. Let the restriction of B to the first $|w_j|$ places in I_j be equal to w_j.

Now let v_j be the initial segment of B of length $m_j + 1$, i.e., up to and including I_j. In case $t = t_i$ is computable, nondecreasing and unbounded, for almost all intervals I_j assigned to t, we have $\mathrm{C}^t(v_j) > |v_j|/3$, because otherwise, for some appropriate constant c, the corresponding codes would yield for almost all j that $\mathrm{C}^{2t}(w_j) \leq |v_j|/3 + c \leq |I_j| - 2$. Furthermore, by construction for every such t there is a constant $c_t > 0$ such that for almost all m, there is some interval I_j assigned to t such that $m_j \leq m$ and $c_t m \leq m_j/4$, hence for almost all m the initial segment of B up to m cannot have Kolmogorov complexity of less than $c_t m$.

We omit the routine proof that A and B are Turing-equivalent and that if A was c.e., then B is c.e. as well, where for the latter fact we need the assumption that g can be effectively approximated from below.

Finally, to see that $\mathrm{C}(B \upharpoonright m) \leq^+ 2 \log m$, notice that in order to determine $B \upharpoonright m$ without time bounds it is enough to know on which of the intervals I_j the assigned time bounds t_i terminate before their computation is canceled by g, which requires one bit per interval, plus another one describing the bit of A coded into B at the end of each interval. □

Lemma 15. *Every high degree contains for every computable, nondecreasing and unbounded function h a set B such that for every computable time bound t and almost all m,*

$$\mathrm{C}^t(B \upharpoonright m) \geq^+ \frac{1}{4}m \quad \text{and} \quad \mathrm{C}(B \upharpoonright m) \leq h(m) \cdot \log m.$$

Proof. The argument is similar to the proof of Lemma 14, but now, when considering interval I_j, we diagonalize against the largest running time among $t_0(m_j), \ldots, t_{h(j)-2}(m_j)$ such that the computation of this value requires not more than $g(j)$ steps. This way we ensure – for any computable time bound t – that at the end of almost all intervals I_j compression by a factor of at most $1/2$ is possible, and that within interval I_j, we have compressibility by a factor of at most $1/4$, up to a constant additive term, because interval I_{j-1} was compressible by a factor of at most $1/2$. □

Kummer's gap theorem asserts that any array noncomputable c.e. Turing degree contains a c.e. set A such that there are infinitely many m such that $\mathrm{C}(A \upharpoonright m) \geq 2 \log m$, whereas all c.e. sets in an array computable Turing degree satisfy $\mathrm{C}(A \upharpoonright m) \leq (1 + \varepsilon) \log m$ for all $\varepsilon > 0$ and almost all m. Similarly, Theorem 16, the main result of this section, asserts a dichotomy for the time-bounded complexity of intial segments between high and nonhigh sets.

Theorem 16. *Let A be any c.e. set.*

(i) If A is high, then there is a c.e. set B with $B =_{\mathrm{T}} A$ such that for every computable time bound t there is a constant $c_t > 0$ such that for all m, it holds that $\mathrm{C}^t(B \upharpoonright m) \geq c_t \cdot m$.
(ii) If A is not high, then there is a computable time bound t such that $\mathrm{C}^t(A \upharpoonright m) \leq^+ \log m$.

Proof. The first assertion is immediate from Lemma 14. In order to demonstrate the second assertion, it suffices to observe that for the modulus of convergence s of the c.e. set A there is a computable function f such that $s(m) \leq f(m)$ for infinitely many m. □

As another easy consequence of Lemma 14, we get an alternative proof of the result due to Juedes, Lathrop and Lutz [JLL] that every high degree contains a strongly deep set. We omit details due to lack of space.

References

[BDG] G. Barmpalias, R. Downey and N. Greenberg. K-trivial degrees and the jump-traceability hierarchy. *Proc. Amer. Math. Soc.*, posted on January 22, 2009, PII S 0002-9939(09)09761-5 (to appear in print).

[Ba] J. M. Barzdins. Complexity of programs to determine whether natural numbers not greater than n belong to a computably enumerable set. *Soviet Math. Dokl.*, 9:1251–1254, 1968.

[BD] L. Bienvenu, R. Downey. Kolmogorov Complexity and Solovay Functions. In *Symposium on Theoretical Aspects of Computer Science 2009, STACS 2009*, 147–158, http://drops.dagstuhl.de/opus/volltexte/2009/1810.

[DH] R. Downey, D. Hirschfeldt. Algorithmic Randomness. Manuscript, 2009

[JLL] D. Juedes, J. Lathrop, J. Lutz. Computational depth and reducibility. *Theor. Comput. Sci.*, 132(2):37–70, 1994.

[K] M. Kummer. Kolmogorov complexity and instance complexity of recursively enumerable sets. *SIAM Journal on Computing*, 25:1123–1143, 1996.

[KS] A. Kučera, T. Slaman. Randomness and recursive enumerability. *SIAM Journal on Computing*, 31:199–211, 2001.

[LV] Li M., P. Vitányi. An Introduction to Kolmogorov Complexity and Its Applications, 3rd ed., Springer, 2008.

[M] J. S. Miller. The K-degrees, low for K-degrees and weakly low for K-degrees, *Notre Dame Journal of Formal Logic*, to appear.

[N] A. Nies. *Computability and Randomness.* Oxford University Press, 2008.

[S] R. Solovay. Draft of a paper (or series of papers) on Chaitin's work. Unpublished notes, 215 pages, 1975.