Initial Segment Complexities of Randomness Notions

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Abstract

Schnorr famously proved that Martin-Löf-randomness of a sequence A can be characterised via the complexity of A's initial segments. Nies, Stephan and Terwijn as well as independently Miller showed that a set is 2-random (that is, Martin-Löf random relative to the halting problem K) iff there is no function f such that for all m and all $n > f(m)$ it holds that $C(A(0)A(1)\ldots A(n)) \leq n - m$; before the proof of this equivalence the notion defined via the latter condition was known as Kolmogorov random.

In the present work it is shown that characterisations of this style can also be given for other randomness criteria like strong randomness (also known as weak 2-randomness), Kurtz randomness relative to K, Martin-Löf randomness of PA-incomplete sets, and strong Kurtz randomness; here one does not just quantify over all functions f but over functions f of a specific form. For example, A is Martin-Löf random and PA-incomplete iff there is no A-recursive function f such that for all m and all $n > f(m)$ it holds that $C(A(0)A(1)\ldots A(n)) \leq n - m$. The characterisation for strong randomness relates to functions which are the concatenation of an A-recursive function executed after a K-recursive function; this solves an open problem of Nies.

In addition to this, characterisations of a similar style are also given for Demuth randomness, weak Demuth randomness and Schnorr randomness relative to K. Although the unrelativised versions of Kurtz randomness and Schnorr randomness do not admit such a characterisation in terms of plain Kolmogorov complexity, Bienvenu and Merkle gave one in terms of Kolmogorov complexity

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defined by computable machines.

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1. Introduction

Kolmogorov complexity [13, 18] aims to describe when a set is random in an algorithmic way. Here randomness means that no type of patterns can be exploited by an algorithm in order to generate initial segments of the characteristic function from shorter programs. Randomness notions have been formalised by Martin-Löf [14], Schnorr [23] and others. A special emphasis was put on describing randomness of a set A in terms of the complexity of the initial segments $A(0)A(1)\dots A(n)$. The first important result in that direction was that Schnorr [24] proved that a set A is Martin-Löf random if and only if for almost all n the prefix free Kolmogorov complexity $H(A(0)A(1)\dots A(n))$ of the $(n+1)$-th initial segment is at least n. It is easy to see that the counterpart of this characterisation is that a set A is *not Martin-Löf random* iff there is an A-recursive function f such that $H(A(0)A(1)\dots A(f(m))) \leq f(m) - m$ for all m. In other words, one can find — relative to A — points to witness the non-randomness effectively. It should be noted that the function f has to be taken relative to A and not relative to some fixed oracle B independent of A as the sets 2-generic relative to B are not Martin-Löf random but would not admit a B-recursive function f witnessing the non-randomness in the way just mentioned.

The scope of the present paper is to study the notions of randomness beyond Martin-Löf randomness. These are the relativised versions "Kurtz random relative to K", "Schnorr random relative to K", and "Martin-Löf random relative to K" where K is the halting problem or any other creative set. In addition, the three independently defined notions of "Demuth random", "weakly Demuth random" and "strongly random" are considered. Strong randomness is by some authors considered to be the next counterpart of Kurtz randomness, although it is not the relativised version; therefore they call Kurtz random also "weakly random" and strongly random also "weakly 2-random" (see, for example, Nies [18]). Strong randomness [12, 22] has various nice characterisations, in particular the following: A is strongly random iff A is Martin-Löf random and forms a minimal pair with K with respect to Turing reducibility [6, Footnote 2]. For these notions, in order to quantify the degree of non-randomness of a sequence, one studies from which value $f(m)$ onwards all initial segments can be compressed by m bits. That is, one looks at functions f such that $C(A(0)A(1)\dots A(n)) \leq n - m$ for all $n > f(m)$; here f might also be an upper bound of the least possible point with this property as one might want to have that f is in a certain Turing degree.

Looking for this type of characterisation for randomness notions between Martin-Löf randomness and 2-randomness (that is, Martin-Löf randomness relative to K) is quite natural. This is because we know [15, 19] that 2-randomness coincides with "Kolmogorov randomness", a notion that is defined by the absence of any f as above.

The results are summarised in the following list:

- A is not 2-random iff there is $f \leq_T A \oplus K$ such that for all m and all $n > f(m)$ it holds that $C(A(0)A(1)\ldots A(n)) \leq n - m$;
- A is not strongly random iff there are $f \leq_T A$ and $g \leq_T K$ such that for all m and all $n > f(g(m))$ it holds that $C(A(0)A(1)\ldots A(n)) \leq n - m$;
- A is not Kurtz random relative to K iff there is $f \leq_T K$ such that for all m and all $n > f(m)$ it holds that $C(A(0)A(1)\ldots A(n)) \leq n - m$;
- A is not Schnorr random relative to K iff there is $f \leq_T K$ such that for infinitely many m and all $n > f(m)$ it holds that $C(A(0)A(1)\ldots A(n)) \leq n - m$;
- A is not Demuth random iff there is an ω-r.e. function f such that for infinitely many m and all $n > f(m)$ it holds that $C(A(0)A(1)\ldots A(n)) \leq n - m$;
- A is not weakly Demuth random iff there are $f \leq_T A$ and an ω-r.e. function g such that for all m and all $n > f(g(m))$ it holds that $C(A(0)A(1)\ldots A(n)) \leq n - m$;
- $A \geq_T K$ or A is not Martin-Löf random iff there is an A-recursive function f such that for all m and all $n > f(m)$ it holds that $C(A(0)A(1)\ldots A(n)) \leq n - m$.
- A is not strongly Kurtz random iff there is a recursive function f such that for all m and all $n > f(m)$ it holds that $C(A(0)A(1)\ldots A(n)) \leq n - m$.

So the main differences between these characterisations are how the bound-function is formed and whether the compressibility condition holds for infinitely many or for all m. Note that due to finite modifications of f it would be equivalent to postulate the condition for all m or for almost all m. Several proofs make use of this fact.

Although the unrelativised versions of Kurtz randomness and Schnorr randomness do not admit such a characterisation in terms of plain Kolmogorov complexity, Bienvenu and Merkle [1] gave one in terms of Kolmogorov complexity defined by computable machines. There is a close connection between the plain Kolmogorov complexity C and prefix-free Kolmogorov complexity H, which will be formalised in Remark 2. This connection helps to establish many bounds obtained for C also for H.

For the scientific background of this paper, the reader is referred to the usual textbooks on recursion theory [20, 21, 25] and algorithmic randomness [2, 5, 13, 18].

2. Notation and basic definitions

For the sake of completeness, we repeat the definition of Kolmogorov complexity.

Definition 1. *Let M be a Turing machine mapping binary strings to binary strings. Then the (plain) Kolmogorov complexity with respect to M is the function C_M from strings to integers defined by*

$$C_M(\tau) = \min\{|\sigma| : M(\sigma) = \tau\}.$$

By effective codings of Turing machines there exists a machine V which is *optimal* for C, i.e. $C_V \leq C_M + O(1)$ for each machine M. In the following we let C denote C_V for a fixed optimal machine. Similarly, there exists a machine U with prefix-free domain such that $C_U \leq C_M + O(1)$ for each machine M with prefix-free domain. We let H denote C_U for a fixed such machine.

Remark 2. *If $C(x) \leq |x| - 1 - 3m$ with a minimal plain code x^* for x, and if n^* and m^* are minimal prefix-free codes for $n = |x|$ and m, respectively, then some prefix-free machine can use $n^* m^* 0^k 1 x^*$ as a prefix-free code for x, where k is chosen such that $|0^k 1 x^*| = n - 3m$.*

It easily follows that there is a constant c such that whenever a set A and a function f satisfy that $C(A(0)A(1)\dots A(n)) \leq n - 3m$ for all m and all $n > f(m)$, then A and f also satisfy that for all $m > c$ and all $n > f(m)$ it holds that $H(A(0)A(1)\dots A(n)) \leq n + H(n) - m$.

We will also use the following theorem.

Theorem 3 (Chaitin's Counting Theorem [3]). *There is a constant c such that for all n and m it holds that*

$$|\{\sigma : |\sigma| = n + 1 \wedge H(\sigma) \leq n + H(n) - m\}| \leq 2^{n-m+c}.$$

Throughout the paper, K denotes the halting problem. f_s is then the s-th approximation of a K-recursive function f, the mapping $x, s \mapsto f_s(x)$ is recursive in both inputs.

An open r.e. class V_e consists of sets A such that for each member $A \in V_e$ it is verified in some finite time s that A belongs to V_e; let $V_{e,s}$ be the class of all A such that it is verified in time s that A belongs to V_e. Now the notion is chosen such that whenever $A \in V_{e,s}$ and $B(m) = A(m)$ for all $m \leq s$ then $B \in V_{e,s}$ as well. An open r.e. class V_e is called finitely generated iff there is a step-number s such that $V_{e,s} = V_e$.

3. Characterising strong randomness and randomness concepts relative to the halting problem

The following notion was originally introduced by Kurtz [12] and is one of the central notions of this paper.

Definition 4 (Kurtz [12]). *A set A is called* strongly random *iff there is no uniform sequence $V_0, V_1, V_2, \ldots$ of uniformly r.e. open classes such that $\mu(V_e) \to 0$ for $e \to \infty$ and $A \in \bigcap_e V_e$.*

Nies [18, Problem 3.6.23] asks whether one can characterise strong randomness via the growth of the initial segment complexity. In the present paper, an answer will be provided, but for that answer the growth-rate depends also on the Turing degree of the set A for which it is asked whether it is strongly random. After the characterisation in Theorem 5, it will be shown in two further results that there is no obvious way to simplify the characterisation.
The next result gives a characterisation of strong randomness in the desired form.

Theorem 5. *The following are equivalent for a set A.*
(a) A is not strongly random.
(b) There is an A-recursive function f and a K-recursive function g such that for all m and all $n \geq f(g(m))$ it holds that $C(A(0)A(1)\ldots A(n)) \leq n - m$.
(c) There is an A-recursive function f and a K-recursive function g such that for all m and all $n \geq f(g(m))$ it holds that $H(A(0)A(1)\ldots A(n)) \leq n + H(n) - m$.

Proof. $(a) \Rightarrow (b)$: Let $V_0, V_1, V_2, \ldots$ be the test which witnesses that A is not strongly random. Now let $h(m)$ be the first index e with $\mu(V_e) \leq 2^{-2m-1}$ and let $h_0, h_1, h_2, \ldots$ be a recursive approximation to h; this approximation is from below, as one can define that $h_0(m) = 0$ and

$$h_{s+1}(m) = \begin{cases} h_s(m) & \text{if } \mu(V_{h_s(m),s}) \leq 2^{-2m-1}; \\ h_s(m) + 1 & \text{otherwise.} \end{cases}$$

Now let $g(m) = \langle m, s \rangle$ for the first s such that $h_s(m) = h(m)$. Next define the A-recursive function f which assigns to $\langle m, s \rangle$ the first encountered $\ell > s + m$ satisfying

$$A(0)A(1)\ldots A(\ell) \cdot \{0,1\}^\infty \subseteq V_{h_s(m)}.$$

Now one defines a plain machine M such that, for all m, n with $n \geq 2m + 1$ and all $x \in \{0,1\}^{n-1-2m}$, $M(1^m 0 x)$ is the x-th string y of length n for which it is verified in time n that $y \cdot \{0,1\}^\infty \subseteq V_{h_n(m)}$; for small n there might be too many of these strings y and then only the first 2^{n-1-2m} of them are in the range of M; but for $n \geq f(g(m))$ it holds that $h_n(m) = h(m)$ and that therefore by the choice of $V_{h(m)}$ there are at most 2^{n-1-2m} of these strings and each of them occurs in the range of M. One of these strings is the prefix of length n of A. Hence, there is a constant c such that for the function $m \mapsto f(g(m+c))$ and every n greater than the value of this function it holds that

$$C(A(0)A(1)\ldots A(n)) \leq n - m.$$

$(b) \Rightarrow (c)$: This follows from Remark 2 and a substitution of g by $\tilde{g}(m) = g(3m)$.

$(c) \Rightarrow (a)$: It follows from Chaitin's Counting Theorem 3 that if ℓ is sufficiently large, then for all n there are at most $2^{n-m+\ell}$ strings σ of length $n+1$ with

$H(\sigma) \leq n + H(n) - m$. Let $g \leq_T K$ and $f = \varphi_e^A$ be the functions from condition (c). Without loss of generality fix them such that g is recursively approximable from below by $g_0, g_1, g_2, \ldots$ and that f is monotone. Now define $V_{\langle m,n,s\rangle}$ as the class of all sets B satisfying one of the following conditions:

1. $\exists t > s[g_t(m) \neq g_s(m)$ or $H_t(n) \neq H_s(n)]$;
2. $\varphi_e^B(g_s(m)) \downarrow > n$;
3. $H(B(0)B(1)\ldots B(n)) \leq n + H_s(n) - m$.

Note that the first condition ensures that *all* sets are enumerated into those classes $V_{\langle n,m,s\rangle}$ where the parameters are not chosen adequately.

The set A is in every class $V_{\langle m,n,s\rangle}$ as whenever the first condition and the second condition do not put A into $V_{\langle m,n,s\rangle}$ then $g_s(m) = g(m)$ and $H_s(n) = H(n)$ and $\varphi_e^A(g(m)) \leq n$ and therefore $H(A(0)A(1)\ldots A(n)) \leq n + H(n) - m$. Furthermore, by taking n sufficiently large (depending on m) and then taking s sufficiently large (depending on m and n) we can make sure that $g_t(m) = g(m)$ and $H_t(m) = H(m)$ for all $t \geq s$ and $\varphi_e^B(g(m)) \geq n$ only for a class of B of measure less than 2^{-m}. It follows then that $\mu(V_{\langle m,n,s\rangle})$ is at most $2^{-m} + 2^{\ell-m}$ as the first condition of putting oracles B into $V_{\langle m,n,s\rangle}$ does not apply, the second condition contributes a class of oracles with measure 2^{-m} and the third condition contributes a class of oracles with measure $2^{\ell-m}$. As ℓ is a constant, one can come as close to measure 0 as desired by starting off with a sufficiently large m and then choosing n in dependence of m and s in dependence on m, n as indicated.

From this sequence of the $V_{\langle m,n,s\rangle}$, one can construct a new sequence of the form $e \mapsto \cap_{n\leq e,m\leq e,s\leq e} V_{\langle m,n,s\rangle}$ which satisfies that the measures of the members tend to 0 and that each member contains the set A as an element. Hence this sequence witnesses that A is not strongly random. □

Note that in the above construction the machine M can be chosen such that its domain is recursive, that is, M can be chosen as a decidable machine. Hence, one could also consider a third equivalent condition which says that A is strongly random iff there is no decidable plain machine U, no K-recursive function g and no A-recursive function f such that for all m and all $n > f(g(m))$ the complexity based on U of $A(0)A(1)\ldots A(n)$ is below $n - m$.

The above conditions (b) and (c) contain a function which is a concatenation of an A-recursive and a K-recursive function. One might ask whether this condition could be simplified by taking only a K-recursive or only an $(A \oplus K)$-recursive function. The answer is "no" as these two choices will give rise to other randomness notions as shown in the next two results.

Definition 6 (Martin-Löf [14]). *A set A is called* Martin-Löf random *iff there is no sequence $V_0, V_1, V_2, \ldots$ of uniformly r.e. open classes such that $\mu(V_e) \leq 2^{-e}$ and $A \in \bigcap_e V_e$.*

Theorem 7. *The following are equivalent for every set A:*
(a) A is not Martin-Löf random relative to K;

(b) *There is* $f \leq_T A \oplus K$ *such that* $\forall m \forall n > f(m)$ $[C(A(0)A(1)\dots A(n)) \leq n - m]$*;*
(c) *There is* $f \leq_T A \oplus K$ *such that* $\forall m \forall n > f(m)$ $[H(A(0)A(1)\dots A(n)) \leq n + H(n) - m]$.

PROOF. If A is Martin-Löf random relative to K then by Miller [15, Theorem 1] for every m there are infinitely many n such that $C(A(0)A(1)\dots A(n)) > n - m$, and by Miller [16, Theorem 4.1] for every m there are infinitely many n such that $H(A(0)A(1)\dots A(n)) > n + H(n) - m$. Hence each of (b) and (c) implies (a).

Assume that (a) holds. Let U^K be a prefix-free universal machine relative to the oracle K and $x, s \mapsto U_s(x)$ be a recursive approximation to this machine such that every U_s is prefix-free. Now there is an $A \oplus K$-recursive function which produces for every m a number $f(m)$ such that there exists z with $|z| + 2m < |U^K(z)| \leq f(m)$, $U^K(z)$ is a prefix of A and $U_s(z) \downarrow = U^K(z)$ for all $s \geq f(m)$.

Now one can construct a plain machine $\tilde{U}$ which sends every input of the form xy with $x \in \mathrm{dom}(U_{|xy|})$ to $U_{|xy|}(x) \cdot y$ and which is undefined on inputs which cannot be brought into this form; note that because of prefix-freeness for each input u the splitting into xy is unique or does not exist. Now for all m there is a z as above. If $U^K(z) = A(0)A(1)\dots A(k)$, then it follows that $\tilde{U}(zA(k+1)A(k+2)\dots A(n)) = U_{n+1}(z) \cdot A(k+1)\dots A(n) = A(0)A(1)\dots A(n)$ and hence $C(A(0)A(1)\dots A(n)) \leq (k - 2m) + (n - k) + O(1) \leq n - m$ for almost all m and all $n > f(m)$. Note that we can modify f for finitely many m such that f satisfies the condition (b). Remark 2 establishes that (c) follows from (b). □

The next result characterises Kurtz randomness relative to K.

Definition 8. *A set A is called* Kurtz-random *iff it is contained in every r.e. class of Lebesgue measure 1.*

Theorem 9. *The following are equivalent for every set A:*
(a) *A is not Kurtz random relative to K;*
(b) *There is a sequence of finitely generated r.e. open classes such that each class contains A and the infimum of their measures is* 0*;*
(c) *There is a K-recursive function f such that for all m and all $n > f(m)$ it holds that $C(A(0)A(1)\dots A(n)) \leq n - m$;*
(d) *There is a K-recursive function f such that for all m and all $n > f(m)$ it holds that $H(A(0)A(1)\dots A(n)) \leq n + H(n) - m$.*

PROOF. (a) $\Rightarrow$ (b): By definition, A is covered by a K-recursive Kurtz-test. According to Bienvenu and Merkle [1, Definition 7] a (K-recursive) Kurtz-test is given by a recursive (K-recursive) function f which determines for each m a finite set $D_{f(m)}$ of strings such that for all m, A has a prefix in $D_{f(m)}$ and the measure of the class of all sets B with a prefix in $D_{f(m)}$ is at most 2^{-m}. For the given K-recursive Kurtz test, let $f_0, f_1, f_2, \dots$ be a recursive approximation of the corresponding function f. Now let $V_{\langle m,s\rangle} = \{B : B$ has a prefix in $D_{f_t(m)}$ for some $t \geq s\}$. It is clear that every $V_{\langle m,s\rangle}$ contains A as a prefix of A is in almost all $D_{f_t(m)}$. Furthermore, as the f_t converge, the union of all $D_{f_t(m)}$ with $t \geq s$ is finite and contains only finitely many strings; that is, the r.e. class generated by it is finitely generated. Furthermore, for every m and every sufficiently large s, $f_t(m) = f(m)$ for all $t \geq s$ and hence $V_{m,s}$ has at most measure 2^{-m}.

$(b) \Rightarrow (c)$: Let $V_0, V_1, V_2, \ldots$ be a given sequence of finitely generated r.e. open classes as in condition (b). Let $V_{e,s}$ be the class of all B for which is verified in time s that they belong to V_e; by choice there is for every e an s with $V_{e,s} = V_e$. For every m let $g_s(m)$ be the smallest number e such that $\mu(V_{e,s}) < 2^{-3m-1}$.

This function $g_s(m)$ is always defined as it is bounded by the index $g(m)$ of the first class whose measure is strictly below 2^{-3m-1}. Now let $f(m)$ be the first step s such that $g_s(m) = g(m)$ and $V_{g_s(m),s} = V_{g(m)}$, that is, all sets which are put into $V_{g(m)}$ are already enumerated into it. Observe that $g_t(m) = g(m)$ for all $t \geq f(m)$. Now let $M(1^m 0x)$ be the x-th string y of length $n+1$ found in $V_{g_n(m)}$ where $n = 3m + |x|$. Note that $A(0)A(1)\ldots A(n)$ is in the range of M whenever $n > f(m)$. As the corresponding $1^m 0x$ has the length $(n+1) - 2m$, it follows that $C(A(0)A(1)\ldots A(n)) \leq n - 2m + O(1) \leq n - m$ for almost all m and all $n > f(m)$. Hence, by a suitable finite modification of f one obtains condition (c).

$(c) \Rightarrow (d)$: This follows from Remark 2 and a substitution of f by $\tilde{f}(m) = f(3m)$.

$(d) \Rightarrow (a)$: Again, by the Counting Theorem 3 there is a constant c such that for every n, m there are at most 2^{n-m+c} strings σ of length $n+1$ with $H(\sigma) \leq n + H(n) - m$. Furthermore let f be given as in condition (d); in particular $H(A(0)A(1)\ldots A(f(m)))$ is at most $f(m) + H(f(m)) - m$. The measure of the class of the sets B with the same property is at most 2^{-m-1+c}. It follows that the mapping of m to the class of all B with $H(B(0)B(1)\ldots B(f(m+c))) \leq f(m+c) + H(f(m+c)) - m$ is a Kurtz test relative to K. □

Let A be given such that every A-recursive function is majorised by a K-recursive one. Then the above characterisations show that A is strongly random iff A is Kurtz random relative to K. But this coincidence does not hold in general as 2-generic sets are Kurtz random relative to K but not strongly random. It should also be noted that there is no oracle B such that every set A which is not strongly random satisfies that there is an B-recursive function f with $C(A(0)A(1)\ldots A(n)) \leq n - m$ for all m and all $n > f(m)$. Hence the condition in Theorem 5 cannot be replaced by a class of functions which is independent of the set A analyzed. It should be noted that the characterisation of "Schnorr random relative to K" is quite similar to that one of "Kurtz random relative to K".

Definition 10 (Schnorr [23]). *A set A is called* Schnorr random *iff there is no sequence $V_0, V_1, V_2, \ldots$ of uniformly r.e. open classes such that $\mu(V_e) = 2^{-e}$ and $A \in \bigcap_e V_e$.*

Theorem 11. *The following are equivalent for a set A:*
(a) *A is not Schnorr random relative to K;*
(b) *There is a K-recursive function f such that for infinitely many m and all $n > f(m)$ it holds that $C(A(0)A(1)\ldots A(n)) \leq n - m$;*
(c) *There is a K-recursive function f such that for infinitely many m and all $n > f(m)$ it holds that $H(A(0)A(1)\ldots A(n)) \leq n + H(n) - m$.*

PROOF. $(a) \Rightarrow (b)$: Downey and Griffiths [7] showed that a set A is not Schnorr random iff there is a recursive sequence of strings $\sigma_0, \sigma_1, \sigma_2, \ldots$ such that infinitely many of these strings are prefixes of A and $\sum_j 2^{-|\sigma_j|}$ is a finite rational number; without loss of generality let the sum be 1. This characterisation can be relativised to K by taking the sequence to be K-recursive. Now one can choose a K-recursive sequence $n_0, n_1, \ldots$ of indices such that for each m it holds that $\sum_{\ell \geq n_m} 2^{-|\sigma_\ell|} \leq 2^{-3m}$; this n_m can be found as the first number with $\sum_{\ell < n_m} 2^{-|\sigma_\ell|} > 1 - 2^{-3m}$. Note that the measure of each subsum $\sum_{\ell = n_m, n_m+1, \ldots, n_{m+1}} 2^{-|\sigma_\ell|}$ is also bounded by 2^{-3m}. Now one can define a plain machine M such that $M(1^m 0 \tau)$ is the τ-th string of length $|\tau| + 3m$ which extends one of the finitely many strings $\sigma^t_{n_m}, \sigma^t_{n_m+1}, \ldots, \sigma^t_{n_{m+1}}$, where $t = |\tau| + 3m$ and σ^s_n is the value of σ_n after s steps in some recursive approximation of the sequence. When approximating n_m, n_{m+1} and the strings $\sigma_{n_m}, \sigma_{n_m+1}, \ldots, \sigma_{n_{m+1}}$, there is a K-recursive function f such that $f(m)$ is an upper bound on the time which is necessary to converge to the correct values; furthermore, one can choose $f(m)$ to be also an upper bound on $|\sigma_\ell| + 3m$ for each of these strings. It follows that for each string η of length at least $f(m)$ there is a string τ of length $|\eta| - 3m$ such that $M(1^m 0 \tau) = \eta$; hence the plain Kolmogorov complexity of all of these strings η is at most $|\eta| + c - 2m$ for some constant c. As there are infinitely many m such that one of the σ_ℓ with $n_m \leq \ell \leq n_{m+1}$ is a prefix of A, it follows that there are infinitely many m such that for all $n \geq f(m)$ it holds that $C(A(0)A(1)\ldots A(n)) \leq n - m$.

$(b) \Rightarrow (c)$: This follows from Remark 2 and a substitution of f by $\tilde{f}(m) = f(3m)$.

$(c) \Rightarrow (a)$: Let $S^{c'}_m = \{B : H(B(0)\ldots B(f(m+c'))) \leq f(m+c') + H(m+c') - m\}$. The Counting Theorem 3 yields a c' such that $(S^{c'}_m)_{m \in \omega}$ can be enlarged to a total Solovay test (as defined by Downey and Griffiths [7]) relative to K. This test covers A, so A is not Schnorr random relative to K. □

4. Characterising Demuth randomness

Demuth has defined in the context of analysis a randomness notion which was formalised as follows in the framework of algorithmic randomness [18, Definition 3.6.24].

Definition 12. *In the following let $V_0, V_1, V_2, \ldots$ be an acceptable numbering of all r.e. open classes. Now one says that a set A is Demuth random iff there is no ω-r.e. function f such that $\mu(V_{f(m)}) \leq 2^{-m}$ for all m and $A \in V_{f(m)}$ for infinitely many m.*

Theorem 13. *The following are equivalent for a set A:*
(a) *A is not Demuth random;*
(b) *There exist ω-r.e. functions g and h such that $A \in V_{g(m),h(m)}$ for infinitely many m and $\mu(V_{g(m),h(m)}) \leq 2^{-m}$ for all m;*
(c) *There exists an ω-r.e. function k such that for infinitely many m and all $n \geq k(m)$ it holds that $C(A(0)A(1)\ldots A(n)) \leq n - m$;*
(d) *There exists an ω-r.e. function $\tilde{k}$ such that for infinitely many m and all $n \geq \tilde{k}(m)$ it holds that $H(A(0)A(1)\ldots A(n)) \leq n + H(n) - m$.*

Proof. $(a) \Rightarrow (b)$: Let f be the ω-r.e. function witnessing that A is not Demuth random. Now define a function $\tilde{h}(e,m)$ such that $\tilde{h}(e,m)$ is the maximum step $s > 0$ for which there is $\ell \in \{1, 2, \ldots, 2^m - 1\}$ with $\mu(V_{e,s-1}) \leq \ell \cdot 2^{-m} < \mu(V_{e,s})$; if no such step exists then $\tilde{h}(e,m) = 0$ and $V_{e,0} = \emptyset$. Note that $\mu(V_e) - \mu(V_{e,\tilde{h}(e,m)}) \leq 2^{-m}$. Given $f, \tilde{h}$, consider a function g such that

$$V_{g(m)} = \bigcup_{\ell = 0,1,\ldots,m,m+1} (V_{f(\ell),\tilde{h}(f(\ell),2m+4-\ell)} - V_{f(\ell),\tilde{h}(f(\ell),2m+2-\ell)})$$

and the function h defined by

$$h(m) = \max\{\tilde{h}(f(\ell), 2m+4-\ell) : \ell \in \{0, 1, \ldots, m, m+1\}\}.$$

Without loss of generality we may assume $V_{g(m)} = V_{g(m),h(m)}$. Furthermore,

$$\mu(V_{f(\ell),\tilde{h}(f(\ell),2m+4-\ell)} - V_{f(\ell),\tilde{h}(f(\ell),2m+2-\ell)}) \leq \mu(V_{f(\ell)} - V_{f(\ell),\tilde{h}(f(\ell),2m+2-\ell)}) \leq 2^{\ell-2m-2}$$

and therefore $\mu(V_{g(m)}) \leq 2^{-m-1} + 2^{-m-2} + \ldots + 2^{-2m-2} \leq 2^{-m}$. It remains to show that g and h are ω-r.e. and that $A \in V_{g(m)} = V_{g(m),h(m)}$. As f and $\tilde{h}$ are both ω-r.e. and $\tilde{h}(e,m)$ makes at most 2^m mind changes, the functions g and h are also ω-r.e. functions. Now consider any i. Then there is $j > i+1$ such that $A \in V_{f(j)}$. It follows that there is an $m \geq j-1$ such that $A \in V_{f(j),\tilde{h}(f(j),2m+4-j)} - V_{f(j),\tilde{h}(f(j),2m+2-j)}$; the reason is that $\mu(V_f(j)) \leq 2^{-j}$ and thus $\tilde{h}(f(j), 2m+2-j) = 0$ for $m \leq j-1$. Now $V_{g(m)}$ contains A and $m > i$. Hence there are infinitely many m with $A \in V_{g(m),h(m)}$. So (b) holds.

$(b) \Rightarrow (c)$: Let g, h be given as required in (b) and assume that $g_s(m) \neq g_{s+1}(m) \vee h_s(m) \neq h_{s+1}(m)$ implies that $h_{s+1}(m) \geq s+1$. Otherwise one can without loss of generality modify g and h accordingly while preserving (b). Now let $M(1^m 0x)$ be the x-th string found in $\{0,1\}^s$ such that $s = |x| + 3m$ and $M(1^m 0x) \cdot \{0,1\}^\infty \subseteq V_{g_s(3m),s} \cup V_{g_s(3m+1),s} \cup V_{g_s(3m+2),s}$. For infinitely many m and all $n > \max\{h(3m), h(3m+1), h(3m+2)\}$ it holds that $A(0)A(1)\ldots A(n) \cdot \{0,1\}^\infty \subseteq V_{g(3m)} \cup V_{g(3m+1)} \cup V_{g(3m+2)}$. For such m, n there are only 2^{n+1-3m} strings of length n qualifying for the search condition, hence there is an x of length $n+1-3m$ such that $M(1^m 0x) = A(0)A(1)\ldots A(n)$ and — if m is furthermore large enough — $C(A(0)A(1)\ldots A(n)) \leq n - m$. Hence one can choose k to be a finite variant of the ω-r.e. function $m \mapsto \max\{h(3m), h(3m+1), h(3m+2)\} + 1$ in order to satisfy condition (c).

$(c) \Rightarrow (d)$: This follows from Remark 2 by choosing $\tilde{k}(m) = k(3m)$.

$(d) \Rightarrow (a)$: Let $\tilde{k}$ be as in condition (d). There is a function f defining the class $V_{f(m)} = \{B : H(B(0)B(1)\ldots B(\tilde{k}(2m))) \leq \tilde{k}(2m) + H(\tilde{k}(2m)) - 2m\}$.

Note that $V_{f(m)}$ has at most measure 2^{-m} for almost all m and we can assume that $V_{f(m)}$ contains A for infinitely many m (otherwise we can replace $\tilde{k}$ by the function $n \mapsto \tilde{k}(n+1)$). Furthermore, there is a recursive function which maps each triple (m, a, b) to an index for the class $\{B : H(B(0)B(1)\ldots B(a)) \leq a + b - 2m\}$ and therefore maps $(m, \tilde{k}(2m), H(\tilde{k}(2m)))$ to $f(m)$. There is a recursive function h such that the approximation of $\tilde{k}(m)$ makes at most $h(m)$ mind changes. As one can code m and the number of mind changes in order to get $\tilde{k}(m)$, for almost all m, the value $H(\tilde{k}(m))$ is at most $h(m)+m$ and once the value $\tilde{k}(m)$ has stabilised, $H(\tilde{k}(m))$ can be approximated from above with $h(m) + m$ many mind changes. It follows that the mapping $m \mapsto (\tilde{k}(2m), H(\tilde{k}(2m)))$ is ω-r.e. with the number of mind changes bounded by $(h(2m) + 2m)^2$ for almost all m. Hence the function f can be taken to be ω-r.e. as well. Then, after a finite modification which preserves f to be ω-r.e., one has that not only for almost all m but indeed for all m the measure of $V_{f(m)}$ is bounded by 2^{-m}. So A is not Demuth random. □

5. Characterising weak Demuth randomness

Demuth [4] also introduced a modified version of Demuth randomness where coverage of a set A by a test requires coverage of A by (almost) all components of the test.

Definition 14. *Given a fixed acceptable numbering $V_0, V_1, \ldots$ of all r.e. open classes, we say that a set A is weakly Demuth random iff there is no ω-r.e. function h such that $\mu(V_{h(m)}) \leq 2^{-m}$ for all m and $A \in \bigcap_m V_{h(m)}$.*

We now prove a characterisation in the same style as before for weak Demuth randomness. The proof of the forward direction will be similar to the proof of that direction for strong randomness, while the backward direction will be a mixture of the proofs for the backward directions for strong and for Demuth randomness.

Theorem 15. *The following are equivalent for a set A:*
(a) A is not weakly Demuth random;
(b) There exists an $f \leq A$ and an ω-r.e. function g such that for all m and all $n \geq f(g(m))$ it holds that $C(A(0)A(1)\ldots A(n)) \leq n - m$;
(c) There exists an $f \leq A$ and an ω-r.e. function g such that for all m and all $n \geq f(g(m))$ it holds that $H(A(0)A(1)\ldots A(n)) \leq n + H(n) - m$.

Proof. $(a) \Rightarrow (b)$: Let $V_{h(0)}, V_{h(1)}, V_{h(2)}, \ldots$, where h is an ω-r.e. function, be a weak Demuth test covering A. Let $h_n(m)$ denote the value of the ω-r.e. approximation to $h(m)$ after its n-th mind change, if that value exists. Let

$g(m) = \langle m, s \rangle$ where s is the number of mind changes in the approximation to $h(2m+1)$. It is obvious that g is ω-r.e.

Next define the A-recursive function f which assigns to $\langle m, s \rangle$ the first encountered $\ell(m)$ satisfying

$$A(0)A(1)\ldots A(\ell(m)) \cdot \{0,1\}^\infty \subseteq V_{h_s(2m+1)}.$$

We define a plain machine M such that, for all m, n with $n \geq 2m+1$ and all $x \in \{0,1\}^{n-2m-1}$, $M(1^m 0 x)$ is the x-th string y of length n for which it is verified in time n that $y \cdot \{0,1\}^\infty \subseteq V_{h_n(m)}$; for small n there might be too many of these strings y and then only the first 2^{n-2m-1} of them are in the range of M.

But if $n > f(g(m)) = \ell(m)$ then it holds that

$$A(0)A(1)\ldots A(n-1) \cdot \{0,1\}^\infty \subseteq A(0)A(1)\ldots A(\ell(m)) \cdot \{0,1\}^\infty \subseteq V_{h(2m+1)}$$

and since $V_{h(2m+1)}$ has measure at most 2^{-2m-1} there can be at most 2^{n-2m-1} strings of length n that have that property. Machine M now witnesses that these strings can be described with codes of length $m+1+(n-2m-1) = n-m$. Since $A(0)A(1)\ldots A(n-1)$ is one of these strings we have achieved the desired code length.

Finally, we need to take care of the constant code length c for coding M. This is achieved by replacing $g(m)$ by $g(m+c)$.

$(b) \Rightarrow (c)$: This follows from Remark 2 and a substitution of g by $\tilde{g}(m) = g(3m)$.

$(b) \Rightarrow (a)$: For the purpose of exposition we first show this direction only involving plain Kolmogorov complexity, which is slightly easier than the version with the added complication of prefix-free Kolmogorov complexity.

Let f and g be as in condition (c). W.l.o.g. we can assume that f is increasing and then also that g is increasing and that every mind change of the ω-r.e. approximation to g is increasing. Let M be an oracle machine that computes f from the oracle A. By f^X we denote the (possibly partial) function computed by M from the oracle X. Note that $f = f^A$.

We define the r.e. classes

$$V_{m,i} = \{X \mid C(X(0)\ldots X(i)) \leq i - m - 1\} \cup \{X : f^X(g(m)) > i\}.$$

It is clear that A is in every $V_{m,i}$ as either $i < f(g(m))$ so that the second condition puts A into $V_{m,i}$ or $i \geq f(g(m))$ in which case the first condition puts A into $V_{m,i}$.

The task of constructing a weak Demuth test covering A reduces therefore to picking a subsequence among the $V_{m,i}$ that meets the measure condition for weak Demuth tests. To do this we observe that there is a constant c such that for every m and i, the first set in the definition of $V_{m,i}$ has measure at most 2^{-m+1+c} and that the same size bound holds for the second set in the union if we only choose a sufficiently large number $i = i(m)$. Hence for every m and for every large enough $i(m)$ the set $V_{m,i}$ has measure at most 2^{-m+2+c}.

Our weak Demuth test will therefore be $(V_{m,i(m)})_m$ where $i(m)$ will be an ω-r.e. function making the second index large enough to make the measure of

$V_{m,i(m)}$ small enough. Note that it is sufficient to ensure $\mu(V_{m,i(m)}) \leq 2^{-m+2+c}$, since by shifting the test, i.e. replacing the m-th component by the $(m+2+c)$-th component we can then meet the defining measure condition for weak Demuth tests. Note also that the size of the first set in the definition of $V_{m,i}$ was bounded independently of i, so that it suffices to make $\mu(\{X : f^X(g(m)) > i(m)\}) \leq 2^{-m+1+c}$.

It remains to show that there indeed exists such an $i(m)$ that is ω-r.e. This can be seen as follows: Let m be fixed and assume we reached some current values $i_s(m)$ and $g_s(m)$ of the approximations to $i(m)$ and $g(m)$, respectively. We approximate $g(m)$ further while enumerating $V_{m,i_s(m)}$ (with the current guess $g_s(m)$ for $g(m)$). If at any time $\{X : f^X(g_s(m)) > i_s(m)\}$ exceeds the measure bound 2^{-m+1+c} we define $i_{s+1}(m)$ to be the maximum of all (finitely many) values $f^X(g_s(m))$ computed so far and restart enumerating $V_{m,i(m)}$ with the new value $i_{s+1}(m)$. This way we ensure that all those X which we already found to satisfy $f^X(g_s(m)) > i_s(m)$ will not satisfy $f^X(g_s(m)) > i_{s+1}(m)$ any more. Hence we need a *disjoint* class of sets X to exceed the measure bound 2^{-m+1+c} before we have to change $i(m)$ again. Since there are only 2^{m-1-c} such classes, we can change $i(m)$ only that many times as long as $g_s(m)$ stays constant. If at any time we change the approximation to $g(m)$ we also restart enumerating $V_{m,i(m)}$ with the new value $g_s(m)$.

Since we increase $i(m)$ only 2^{m-1-c} times for each intermediate value $g_s(m)$ of $g(m)$, and since g is itself ω-r.e., $i(m)$ is ω-r.e. as well.

$(c) \Rightarrow (a)$: The case for prefix-free Kolmogorov complexity works similar but adds an additional complication, that is, when we enumerate the equivalents of the sets $V_{i,m}$ in the prefix-free case, that is, the sets

$$V'_{i,m} = \{A \mid H(X(0)\dots X(i)) \leq i + H(i) - m - 1 \vee f^X(g(m)) > i\},$$

we can only approximate the exact value of $H(i)$ from above, and when the value of this approximation changes we will also need to cancel (that is, restart the enumeration of) this component of the test. Therefore, more cancelations may occur than before and we need to show that the resulting function $i(m)$ is still ω-r.e.

To see this consider the following inductive argument for every m: Assume that the current approximation $i_s(m)$ to $i(m)$ has been reached with s many mind changes and that we have just canceled the component $V_{m,i_{s-1}(m)}$. We now want to start enumerating $V_{m,i_s(m)}$ and would need $H(i_s(m))$ for this. The value $i_s(m)$ can be described by a finite code describing the whole approximation procedure of $V_{i,m}$ together with s. Therefore $H(i_s(m)) \leq^+ \log s + \log m$ and the approximation of the value of $H(i_s(m))$ can make at most $\log s + \log m$ mind changes. □

6. Characterising Turing-incomplete Martin-Löf random sets

Recall that a set A is PA-complete iff there is an A-recursive consistent and complete extension of Peano Arithmetic. This condition is equivalent to saying that every partial-recursive $\{0,1\}$-valued function has a total A-recursive extension. Stephan [26] showed that a Martin-Löf random set is Turing above K iff it is PA-complete. This showed that the Martin-Löf random sets fall into two classes: those above K which coincide with the PA-complete ones and those not above K which coincide with the PA-incomplete ones. The next result shows that the PA-incomplete Martin-Löf random sets have a natural characterisation in terms of initial segment complexity. Note that all Demuth random and all strongly random sets are PA-incomplete. On the other hand, there are Martin-Löf random sets which are PA-complete like Chaitin's Ω. Gács [10] and Kučera [11] showed that every $\mathbf{a} \geq_T K$ contains a Martin-Löf random set and those are PA-complete.

In a recent article, Franklin and Ng [9] introduced a randomness notion called "difference randomness" that is defined in a way that on an intuitive level resembles the definition of Demuth randomness — and in fact they proved that this notion is characterised by passing all so called strict Demuth tests. They showed that this new randomness notion is equivalent to being Turing-incomplete and Martin-Löf random.

Theorem 16. *The following statements are equivalent for a set A:*
(a) *A is PA-complete or A is not Martin-Löf random;*
(b) *$A \geq_T K$ or A is not Martin-Löf random;*
(c) *There is an A-recursive function f such that $C(A(0)A(1)\ldots A(n)) \leq n-m$ for all m and all $n > f(m)$;*
(d) *There is an A-recursive function f such that $H(A(0)A(1)\ldots A(n)) \leq n + H(n) - m$ for all m and all $n > f(m)$.*

Proof. $(a) \Leftrightarrow (b)$ is already known [26] and $(c) \Rightarrow (d)$ follows from Remark 2.

$(b) \Rightarrow (c)$: If A is not Martin-Löf random, the construction of f is straightforward, using the fact that A has $2m$-compressible prefixes for each m.

If $K \leq_T A$, then if A were Martin-Löf random relative to K, K would be a base for Martin-Löf randomness. By [18, Theorem 5.1.22] we would have $K \in \mathrm{Low(MLR)}$, a contradiction. So A is not Martin-Löf random relative to K and by Theorem 7 there is an $A \oplus K$-recursive function f with

$$\forall m \, \forall n > f(m) \; [C(A(0)A(1)\ldots A(n)) \leq n-m].$$

By assumption, this function f is also A-recursive and satisfies the claim.

$(d) \Rightarrow (b)$: Assume that $A \not\geq_T K$ as otherwise there is nothing to prove. Let f be as in condition (d) and let U be the universal prefix-free machine that defines H. The function $m \mapsto f(2m)$ is A-recursive and does not majorise the function

$$g\colon m \mapsto \max\{U(\tau) : \tau \in \mathrm{dom}(U) \cap \{0,1\}^m\},$$

since $A \not\geq_T K$ and only oracles Turing above K can compute functions which majorise g. Hence there are infinitely many m where the largest value $U(\tau)$ for $\tau \in \mathrm{dom}(U) \cap \{0,1\}^m$ is beyond $f(2m)$. By assumption on f and τ,

$$H(A(0)A(1)\ldots A(U(\tau))) \leq U(\tau) + H(U(\tau)) - 2m \leq U(\tau) + |\tau| - 2m = U(\tau) - m.$$

This shows that A is not Martin-Löf random. □

Stephan and Wu [27] called a set A strongly Kurtz random iff there is no recursive function f such that $H(A(0)A(1)\dots A(f(m))) \leq f(m) - m$ for all m. In the following it is shown that this condition can be generalized in the flavour of the other conditions given here.

Theorem 17. *The following are equivalent for a set A:*
(a) A is not strongly Kurtz-random;
(b) There is a recursive function g such that $C(A(0)A(1)\dots A(n)) \leq n - m$ for all m and all $n > g(m)$;
(c) There is a recursive function h such that $H(A(0)A(1)\dots A(n)) \leq n + H(n) - m$ for all m and all $n > h(m)$.

Proof. (a) ⇒ (b): Let U be a prefix-free universal machine. Now one can define a machine $\tilde{U}$ such that for all $\sigma \in \mathrm{dom}(U)$ and all τ that $\tilde{U}(\sigma\tau) = U(\sigma)\tau$. Note that the prefix σ of $\sigma\tau$ is the unique prefix in the domain of U whenever such a prefix exists; hence $\tilde{U}$ is well-defined. Now $\tilde{U}$ witnesses that $C(U(\sigma)\tau) \leq |\sigma| + |\tau| + c$ for some constant c whenever $U(\sigma)$ is defined. Assuming that f is a recursive function witnessing case (a) we have $H(A(0)A(1)\dots A(f(m+c))) \leq f(m+c) - (m+c)$ for all m. Now it holds that $C(A(0)A(1)\dots A(n)) \leq n-(m+c)+c = n-m$ whenever $n \geq f(m+c)$. Hence choosing $g(m) = f(m+c)$ satisfies condition (b).

(b) ⇒ (c): This follows from Remark 2 and a substitution of f by $\tilde{f}(m) = f(3m)$.

(c) ⇒ (a): Let h be the recursive function witnessing condition (c). It holds that $H(h(2m)) \leq m$ for almost all m and therefore $H(A(0)A(1)\dots A(h(2m))) \leq h(2m) + m - 2m \leq h(2m) - m$ for almost all m. Choosing f as a suitable finite variant of the mapping $m \mapsto h(2m)$ proves this direction. □

7. Conclusion and future work

The overall idea of this article is to measure the degree of randomness of a set A by analyzing the function

$$R^A(m) = \min\{k \in \mathbb{N}\cup\{\infty\} : \forall n\, [k < n < \infty \Rightarrow C(A(0)A(1)\dots A(n)) \leq n-m]\}.$$

Note that $R^A(m) \leq R^A(m+1)$ for all m and that A is 2-random iff R^A assumes the value ∞ on some inputs. One can now reformulate the main results of the paper in terms of the function R^A. For example, A is strongly random iff there are no $f \leq_T A$ and no $g \leq_T K$ such that the concatenation $n \mapsto f(g(n))$ dominates R^A. Here f dominates g iff $f(m) \geq g(m)$ for almost all $m \in \mathbb{N}$. The other results in this article can be formulated analogously in an obvious way. When looking at R^A, one could define a new reducibility as follows.

Definition 18. *A set A is said to be Kurtz-Kolmogorov-reducible to B ($A \leq_{\mathrm{KK}} B$) if there is a recursive function f and a constant c such that for all $m \in \mathbb{N}$ it* *holds that $R^A(m) \leq f(R^B(m+c))$. Here, f is extended to $\mathbb{N} \cup \{\infty\}$ by letting $f(\infty) = \infty$, where the conventions $n \leq \infty$, $\infty \leq \infty$ and $\infty \not\leq n$ hold for all $n \in \mathbb{N}$.*

Note that this definition is invariant under recursive permutations g, so if $B = \{g(n) : n \in A\}$ then $A \equiv_{\mathrm{KK}} B$. Indeed, to see $A \leq_{\mathrm{KK}} B$, one has to choose f such that $f(k) = \max\{g(n) : n \leq k+1\}$ and to observe that $C(B(0)B(1)\dots B(f(k))) \leq C(A(0)A(1)\dots A(k)) + (f(k)-k) + c$ for some constant c. It then follows that $R^B(m) \leq f(R^A(m+c))$ for all m. Using a similar argument one can show that all sets A, B satisfy $A \oplus B \leq_{\mathrm{KK}} A$.

It is clear by definition that C-reducibility (for a definition see for example Downey and Hirschfeldt [5]) implies KK-reducibility. Since rK-reducibility as defined by Downey, Hirschfeldt and LaForte [8] implies C-reducibility, it also implies KK-reducibility. The inverse implications do not hold in general, as for these two reducibilities one can easily come up with sets A and B such that $A \oplus B$ is not reducible to A, while on the other hand this is always true for KK-reducibility, as explained above. It is an open question whether there exist interesting additional conditions on A and B that would ensure that the inverse implications holds.

Besides this, it can be seen that the following classes are closed upward under KK-reducibility (that is, whenever A is in the class and $A \leq_{\mathrm{KK}} B$ then also B is in the class):

- the class of all 2-random sets (as it consists of the greatest KK-degree);
- the class of all strongly Kurtz random sets (as it consists of all degrees except the least one);
- the class of all Demuth random sets;
- the class of all sets which are Kurtz random relative to K;
- the class of all sets which are Schnorr random relative to K.

The reason is that for all of these classes, the randomness notion is defined by comparing the growth rate of R^A with that of a certain list of functions which do not depend on A.

Somehow, for the classes $\{A : A$ is strongly random$\}$, $\{A : A$ is weakly Demuth random$\}$ and $\{A : A$ is Martin-Löf random and $A \not\geq_T K\}$, A becomes involved and the upward closure is no longer guaranteed. It is therefore natural to ask whether the role of A could be replaced by something else, so that one or both of the mentioned classes would be closed upward with respect to KK-reducibility.

For the case of strong randomness, Yu Liang pointed out to us that a result in a recent article [28] of his implies that such a replacement cannot exist. This can be seen as follows: Assume that an appropriate replacement set B existed such that this single, fixed B could be inserted instead of the set A appearing in the initial segment complexity characterisation of strong randomness above.

Then we could verify whether a given set A is not strongly random by evaluating the formula

$$\exists f \leq_T B \, \exists g \leq_T K \, \forall m \, \forall n > f(g(m)) \, [C(A(0)A(1) \dots A(n)) \leq n - m],$$

which can be equivalently expressed as $\exists e \, \exists d \, \forall m \, \forall n \, P(e, d, m, n)$, where P is an appropriate (obviously non-recursive) predicate.

In the Borel hierarchy (see, e.g., [17]), a topological hierarchy where we do not have to pay attention to the computability of the predicates used, this would show that being not strongly random is a $\mathbf{\Sigma}^0_2$ property, making strong randomness a $\mathbf{\Sigma}^0_3$ property. This would be in contradiction to the corresponding result in Yu's article [28].

For the cases of $\{A : A$ is Martin-Löf random and $A \not\geq_T K\}$ and $\{A : A$ is weakly Demuth random$\}$ the question is still open.

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References

[1] Bienvenu, L., Merkle, W.: Reconciling Data Compression and Kolmogorov Complexity. International Colloquium on Automata, Languages and Programming (ICALP 2007). LNCS, vol. 4596, pp. 643–654. Springer, Heidelberg (2007).

[2] Calude, C.S.: Information and Randomness. An Algorithmic Perspective. Second Edition. Springer, Heidelberg (2002).

[3] Chaitin, G.J.: A theory of program size formally identical to information theory. Journal of the ACM 22, 329–340 (1975).

[4] Demuth, O.: Some classes of arithmetical real numbers (in Russian). Commentationes Mathematicae Universitatis Carolinae 23(3), 453–465 (1982).

[5] Downey, R., Hirschfeldt, D.: Algorithmic Randomness and Complexity. Springer, Berlin (2010).

[6] Downey, R., Nies, A., Weber, R., Yu, L.: Lowness and Π^0_2 Nullsets. The Journal of Symbolic Logic 71, 1044–1052 (2006).

[7] Downey, R., Griffiths, E.: On Schnorr randomness. The Journal of Symbolic Logic 69, 533–554 (2004).

[8] Downey, R., Hirschfeldt, D., LaForte, G.: Randomness and reducibility. The Journal of Computer and System Sciences 68, 96-114 (2004).

[9] Franklin, J., Ng, K.: Difference randomness. Proceedings of the AMS 139, 345–360 (2011).

[10] Gács, P.: Every sequence is reducible to a random one. Information and Control 70, 186–192 (1986).

[11] Kučera, A.: Measure, Π^0_1-classes and complete extensions of PA. Recursion Theory Week. LNM, vol. 1141, pp. 245–259. Springer, Heidelberg (1985).

[12] Kurtz, S.A.: Randomness and genericity in the degrees of unsolvability. PhD dissertation, University of Illinois at Urbana-Champaign (1981).

[13] Li, M., Vitányi, P.: An Introduction to Kolmogorov Complexity and Its Applications. Third Edition. Springer, Heidelberg (2008).

[14] Martin-Löf, P.: The definition of random sequences. Information and Control 9, 602–619 (1966).

[15] Miller, J.S.: Every 2-random real is Kolmogorov random. The Journal of Symbolic Logic 69, 907–913 (2004).

[16] Miller, J.S.: The K-degrees, low for K degrees and weakly low for K oracles. Notre Dame Journal of Formal Logic 50, 381–391 (2009).

[17] Moschovakis, Y.N.: Descriptive Set Theory. Second Edition. Mathematical Surveys and Monographs, vol. 155, American Mathematical Society (2009).

[18] Nies, A.: Computability and Randomness. Oxford Science Publications (2009).

[19] Nies, A., Stephan, F., Terwijn, S.A.: Randomness, relativization and Turing degrees. The Journal of Symbolic Logic 70, 515–535 (2005).

[20] Odifreddi, P.: Classical Recursion Theory I. North-Holland (1989).

[21] Odifreddi, P.: Classical Recursion Theory II. Elsevier (1999).

[22] Osherson, D., Weinstein, S.: Recognizing the strong random reals. The Review of Symbolic Logic 1, 56–63 (2008).

[23] Schnorr, C.-P.: Zufälligkeit und Wahrscheinlichkeit. LNCS, vol. 218. Springer, Heidelberg (1971).

[24] Schnorr, C.-P.: Process complexity and effective random tests. Journal of Computer and System Sciences 7, 376–388 (1973).

[25] Soare, R.I.: Recursively Enumerable Sets and Degrees. Springer, Heidelberg (1987).

[26] Stephan, F.: Martin-Löf Random and PA-complete Sets. Proceedings of ASL Logic Colloquium 2002. ASL Lecture Notes in Logic 27, 342–348 (2006).

[27] Stephan, F., Wu, G.: Presentations of K-Trivial Reals and Kolmogorov Complexity. New Computational Paradigms: First Conference on Computability in Europe, CiE 2005. LNCS, vol. 3526, pp. 461–469. Springer, Heidelberg (2005).

[28] Yu, L.: Descriptive set theoretical complexity of randomness notions. Fundamenta Mathematicae 215 , 219–231 (2011).